MECH 401
Mechanical Design Applications
Dr. M. O'Malley – Master Notes

Spring 2008
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Updates

- HW 1 due Thursday (1-17-08)
- Last time
  - Introduction
  - Units
  - Reliability engineering
  - Materials
- This week
  - Load and stress analysis

- Quiz #1 is Jan 29 (in class)
  - Covers material through Chapter 3 (first 2 weeks of class)

Equilibrium

- Basic equations of equilibrium enable determination of unknown loads on a body

- For a body at rest, (recalling from statics)
  - $\Sigma F = 0$ $\Sigma M = 0$
- For a body in motion, (recalling from dynamics)
  - $\Sigma F = ma$ $\Sigma M = I\alpha$

Determining loads

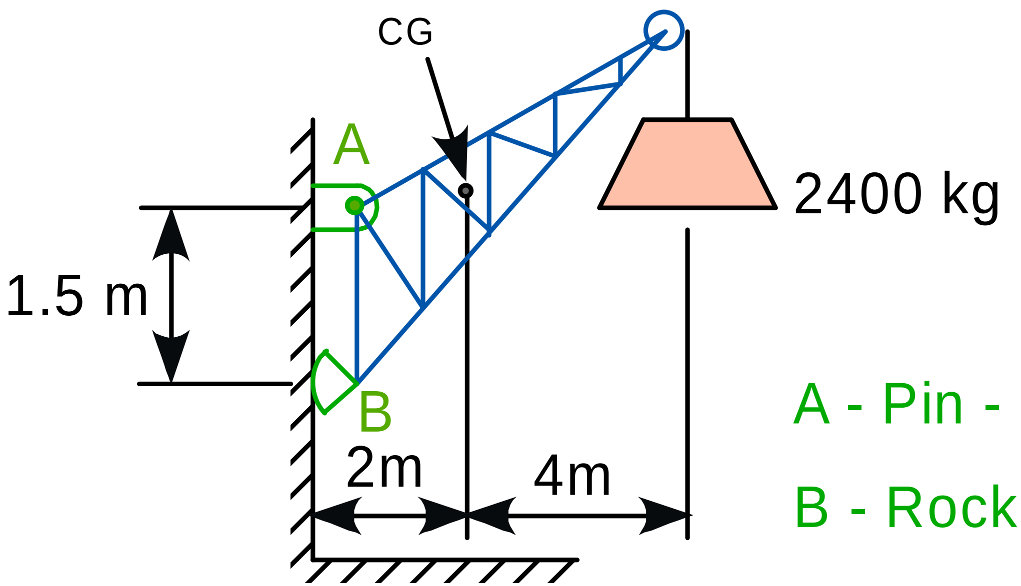
- Machine and structural components are load-carrying members
- We need to be able to analyze these loads in order to design components for the proper conditions
- Determining loads
  - Engines/compressors operate at known torques and speeds (easy!)
  - Airplane structure loads depend on air turbulence, pilot decisions (not so easy)
  - Experimental methods / past performance
- Often we can determine loads by using a free body diagram (FBD)
- Gives a concise view of all the forces acting on a rigid body

Steps for drawing FBD's

1. Choose your body and detach it from all other bodies and the ground – sketch the contour
2. Show all external forces
   - From ground
   - From other bodies
   - Include the weight of the body acting at the center of gravity (CG)
3. Be sure to properly indicate magnitude and direction
   - Forces acting *on* the body, not *by* the body
4. Draw unknown external forces
   - Typically reaction forces at ground contacts
   - Recall that reaction forces constrain the body and occur at supports and connections
5. Include important dimensions

Example – Drawing FBD's

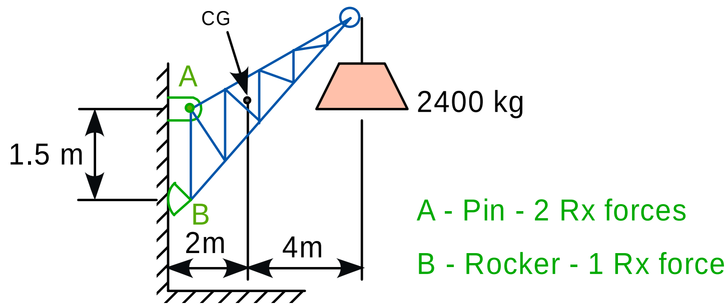
- Fixed crane has mass of 1000 kg
- Used to lift a 2400 kg crate
- Find: Determine the reaction forces at A and B



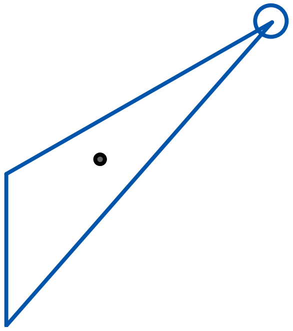
A - Pin - 2 Rx forces

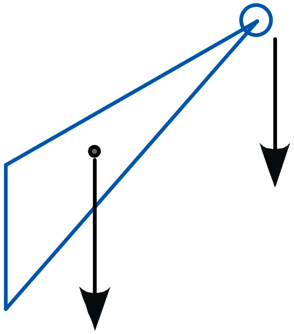
B - Rocker - 1 Rx force

Crane example (FBD's) cont.

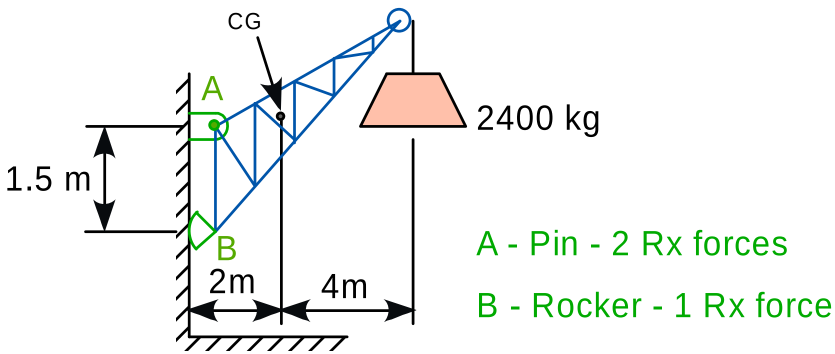


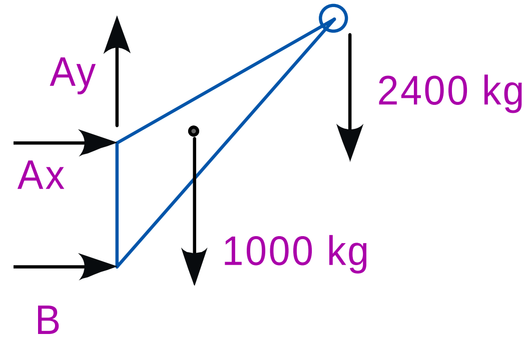
1. Choose your body and detach it from all other bodies and the ground – sketch the contour
2. Show all external forces (from ground, from other bodies). Include weight of the body acting at the center of gravity (CG)
3. Be sure to properly indicate magnitude and direction (acting ON the body, not BY the body)



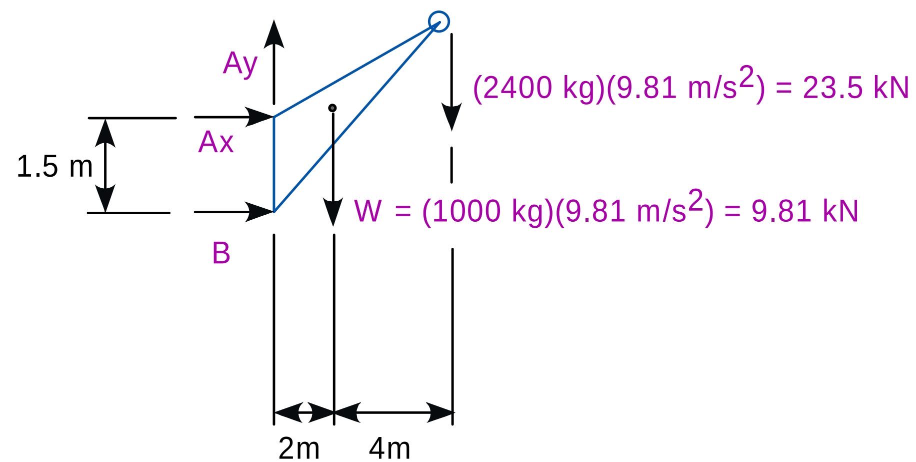


Crane example (FBD's) cont.





4. Draw unknown external forces (typically reaction forces at ground contacts). Recall that reaction forces constrain the body and occur at supports and connections
5. Include important dimensions



Find A_x, A_y, and B

- $\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma M = 0$
- Find B: $+\circlearrowleft \Sigma M_A = 0$
  - $B(1.5) - (9.81)(2) - (23.5)(6) = 0$
  - $B = 107.1$ kN $\longrightarrow$
- Find A_x: $\overset{+}{\longrightarrow} \Sigma Fx = 0$
  - $A_x + B = 0$
  - $A_x = -107.1$ kN
  - $A_x = 107.1$ kN $\longleftarrow$
- Find A_y: $+\uparrow \Sigma Fy = 0$
  - $A_y - 9.81 - 23.5 = 0$
  - $A_y = 33.3$ kN $\uparrow$

Which forces contribute to ΣM_A?

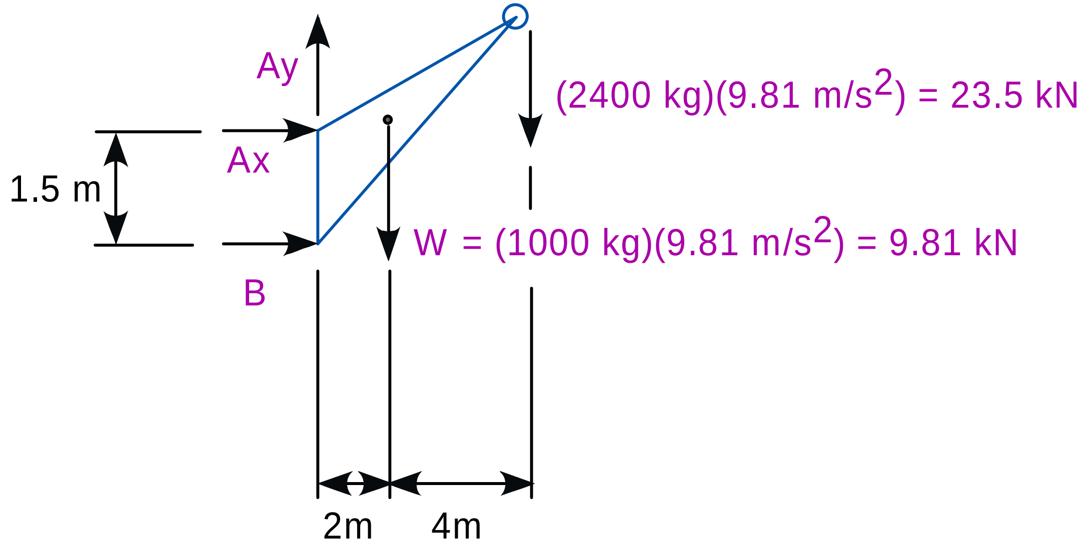
B, 9.81, 23.5

Which forces contribute to ΣF_X?

A_x, B

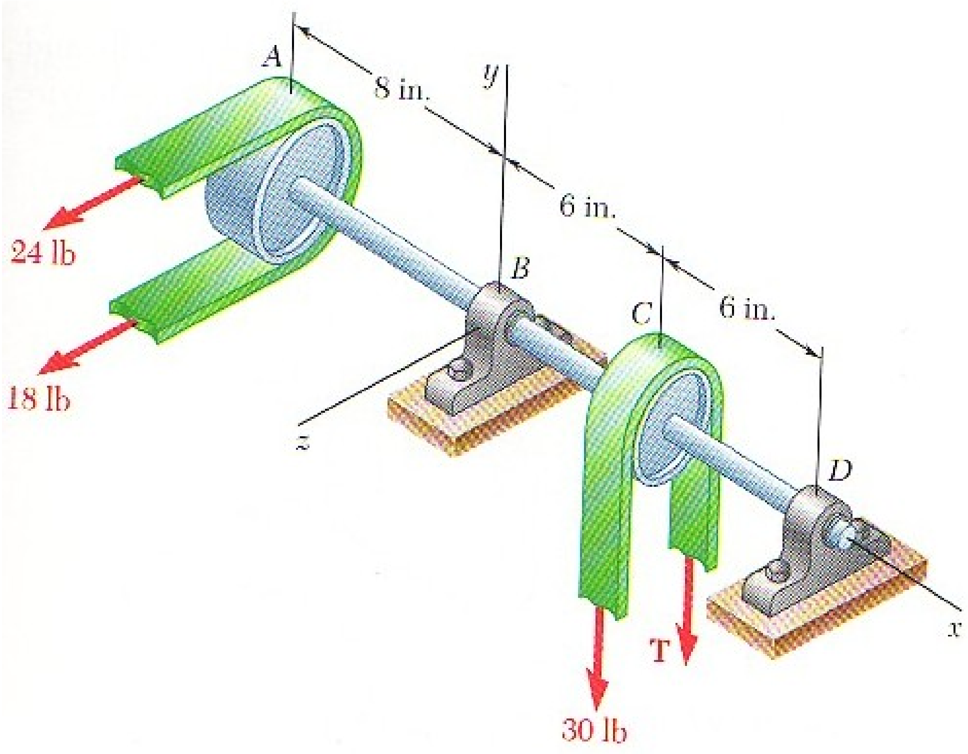
Which forces contribute to ΣF_y?

A_y, 9.81, 23.5



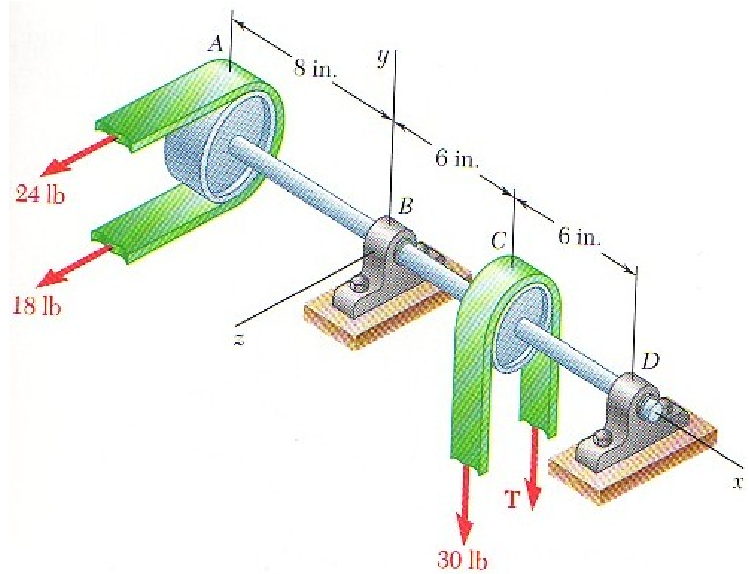
3-D Equilibrium example

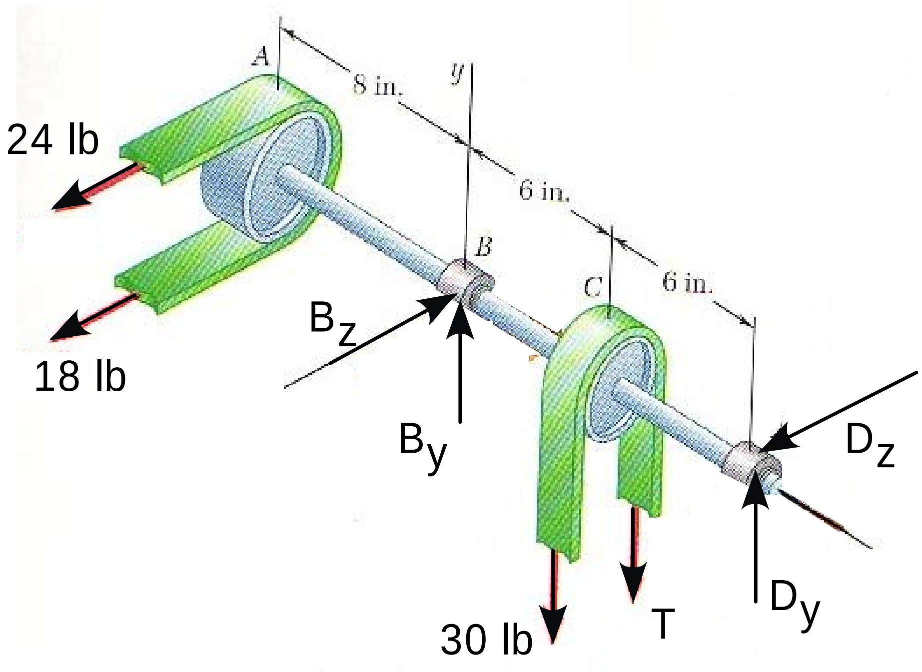
- 2 transmission belts pass over sheaves welded to an axle supported by bearings at B and D
- A radius = 2.5”
- C radius = 2”
- Rotates at constant speed
- Find T and the reaction forces at B, D
- Assumptions:
  - Bearing at D exerts no axial thrust
  - Neglect weights of sheaves and axle



Draw FBD

- Detach the body from ground (bearings at B and D)
- Insert appropriate reaction forces





Solve for reaction forces for each axis

If we sum moments about x (along the shaft), what forces are involved?

24lb, 18lb, 30 lb, T

If we sum moments about y, what forces are involved?

24lb, 18lb, D_z

If we sum moments about z, what forces are involved?

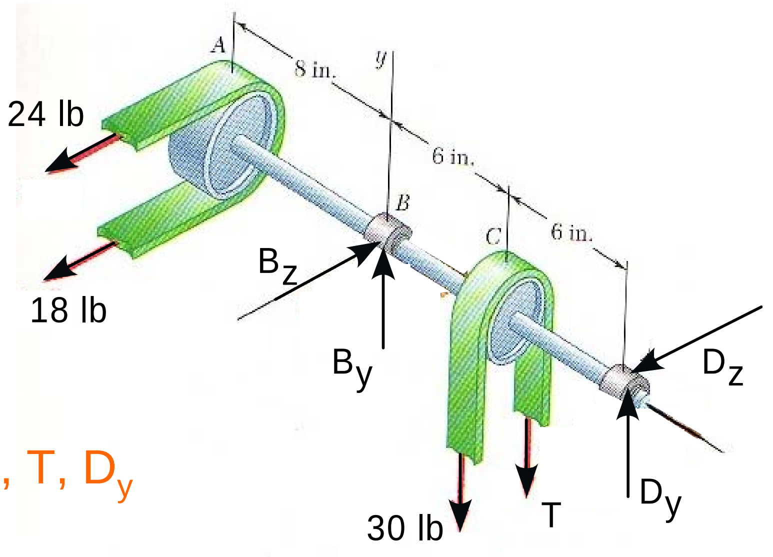
30lb, T, D_y

If we sum forces in y, what forces will we need to consider?

B_y, 30lb, T, D_y

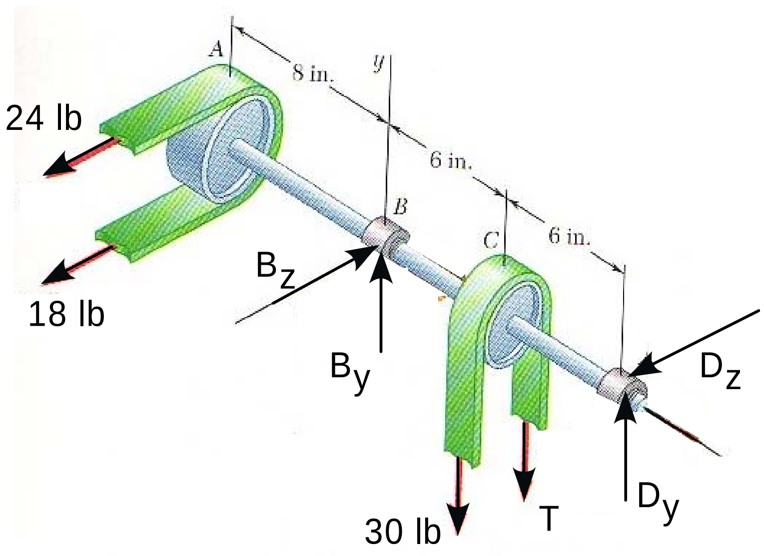
If we sum forces in Z, what forces do we need to consider?

24lb, 18lb, B_z, D_z



Solve for reaction forces for each axis

- $+\circlearrowleft$ $\Sigma M_x = 0 = (24)(2.5) - 18(2.5) + 30(2) - T(2)$
  - $T = 37.5$ lb
- $+\circlearrowleft$ $\Sigma M_y = 0 = (24)(8) + (18)(8) - D_z(12)$
  - $D_z = 28$ lb
- $+\circlearrowleft$ $\Sigma M_z = 0 = -(30)(6) - (37.5)(6) + D_y(12)$
  - $D_y = 33.75$ lb
- $+\uparrow$ $\Sigma F_y = 0 = B_y - 30 - 37.5 + D_y$
  - $B_y + D_y = 67.5$
  - $B_y = 33.75$ lb
- $+\swarrow$ $\Sigma F_z = 0 = 24 + 18 - B_z + D_z$
  - $42 + D_z = B_z$
  - $B_z = 70$ lb



B = (33.75 lb)**j** – (70 lb)**k**

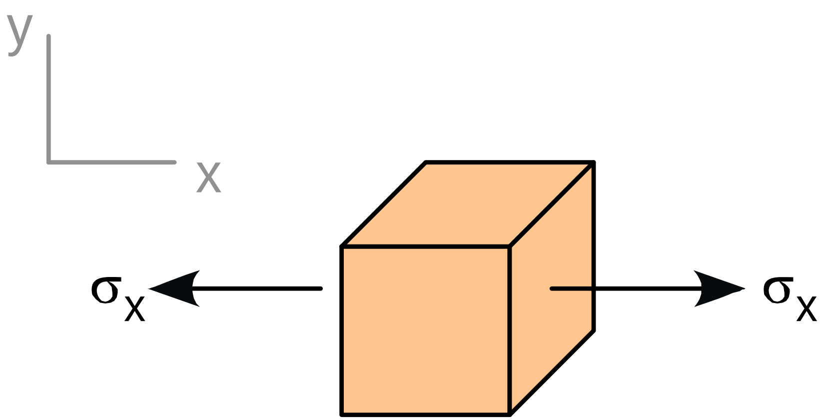
D = (33.75 lb)**j** + (28 lb)**k**

Linearly Elastic Material Behavior

- Some linearly elastic materials:
  - Metals
  - Wood
  - Concrete
  - Plastic
  - Ceramic and glass
- Linearly elastic materials obey Hooke's Laws!

Uniaxial Stress State

- Hooke's Law in uniaxial tension-compression:
  - $\sigma_x = E\varepsilon_x$
- Also, for isotropic and homogenous material
  - Poisson's Ratio
  - $\nu = -\varepsilon_y / \varepsilon_x$
  - 0 (cork) < ν < 0.5 (rubber)



Thermal Stresses

- Expansion of Parts due to temperature
  - without constraint – no stresses
  - with constraint – stress buildup

Expansion of a rod vs. a hole

Differential Thermal Expansion

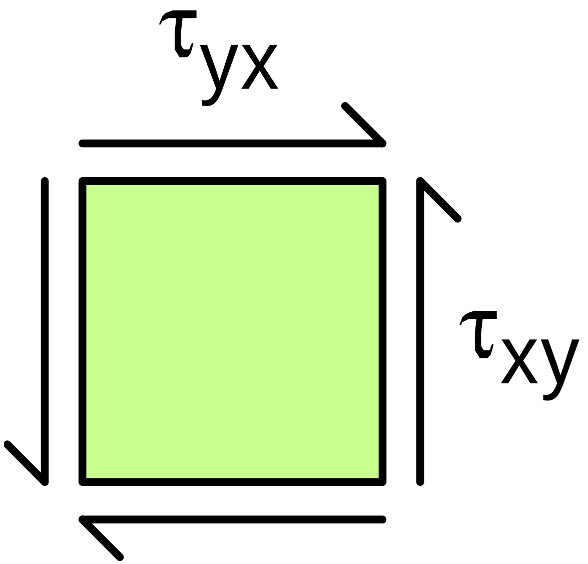
Two material with differential thermal expansion rates that are bound together

Brass and steel

Metals vs. plastic

Hooke's Law for Shear

- $\tau = G\gamma$
- G ?
  - Shear Modulus
  - Modulus of Rigidity
- Note: No equivalent to Poisson for shear (no coupling between axes)

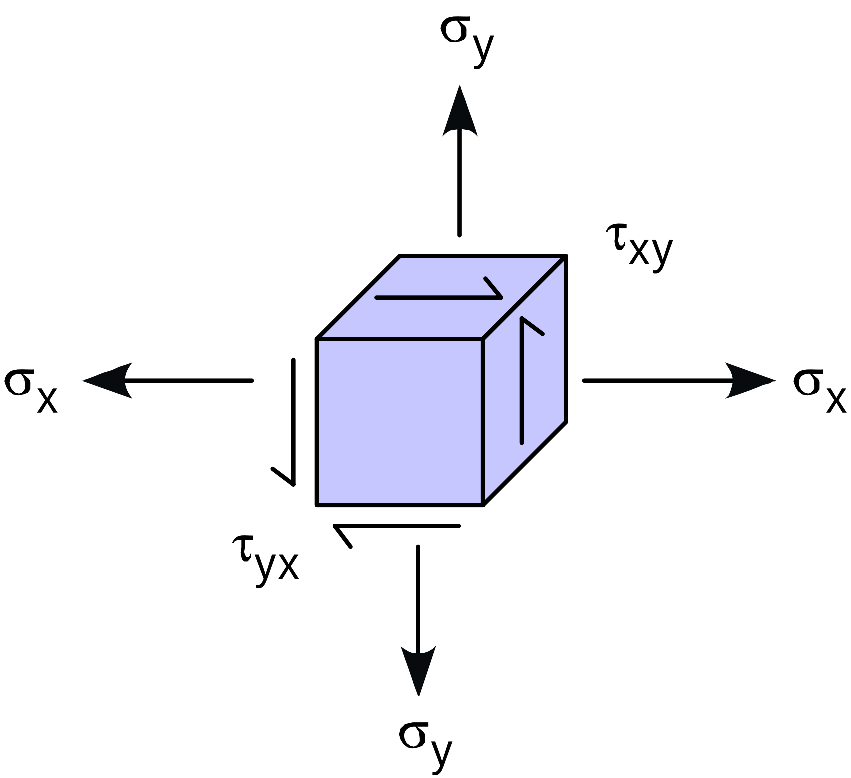


Relating E and G

$$G = \frac{E}{2(1+\nu)}$$

- For linearly elastic, homogenous, isotropic material characterized by TWO independent parameters

Hooke's Law for Biaxial Stress State

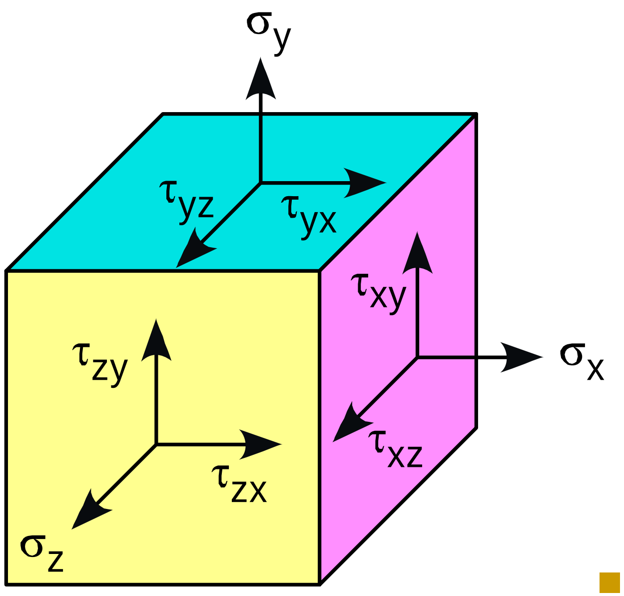


$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Hooke's Law for triaxial state of stress



- Most general case of static loading
- Coupled:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\left(\sigma_y + \sigma_z\right)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}\left(\sigma_x + \sigma_z\right)$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

- Decoupled:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

- Note:

$$G = \frac{E}{2(1+\nu)}$$

More on Hooke's Law

- Used to relate loading to stress state through geometry
- Analytical solutions for classical forms of loading
  - Axial cases:
    - Column in tension (trivial)
    - Column in compression (non-trivial)
      - We'll do this later
  - Other cases:
    - Beam in pure bending
    - Beam in bending and shear
    - Shaft in torsion

Column in tension

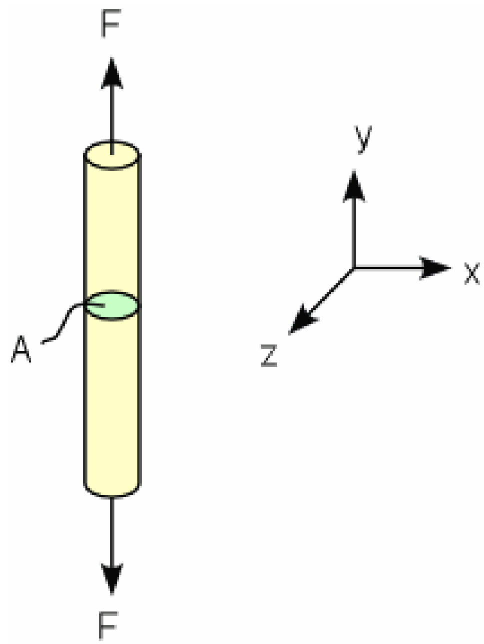
- Uniaxial tension

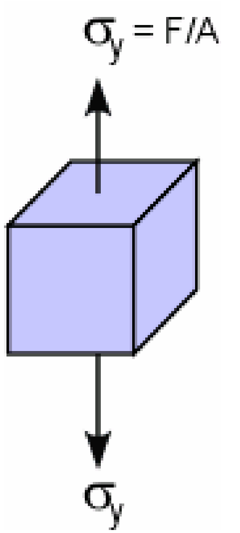
$$\sigma = \frac{F}{A}$$

- Hooke's Law

$$\varepsilon_x = \varepsilon_z = -\frac{\nu F}{EA}$$

$$\varepsilon_y = \frac{\sigma_y}{E} = \frac{F}{EA}$$





Beam in pure bending

- What does PURE mean?
  - Moment is constant along the beam
- Other assumptions
  - Symmetric cross-section
  - Uniform along the length of the beam
  - Linearly elastic
  - Homogenous
    - Constant properties throughout
  - Isotropic
    - Equal physical properties along each axis
- Enable geometric arguments
- Enable the use of Hooke's Law to relate geometry (ε) to stress (σ)

Beam in pure bending

- Result –

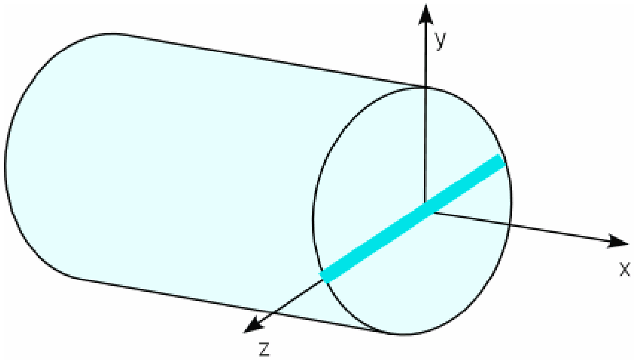
$$\sigma_x = \frac{Mc}{I_z} = \frac{M}{Z}$$

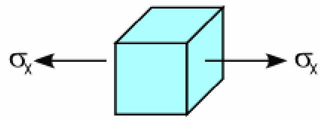
$$Z = \frac{I}{c} \qquad \text{(section modulus)}$$

  - I is the area moment of inertia: $I_z = \int_A y^2 dA$
  - M is the applied bending moment
  - c is the point of interest for stress analysis, a distance (usually y_{max}) from the neutral axis (at y = 0)
    - If homogenous (E = constant), neutral axis passes through the centroid
  - Uniaxial tension:

$$\varepsilon_x = \frac{My}{EI}$$

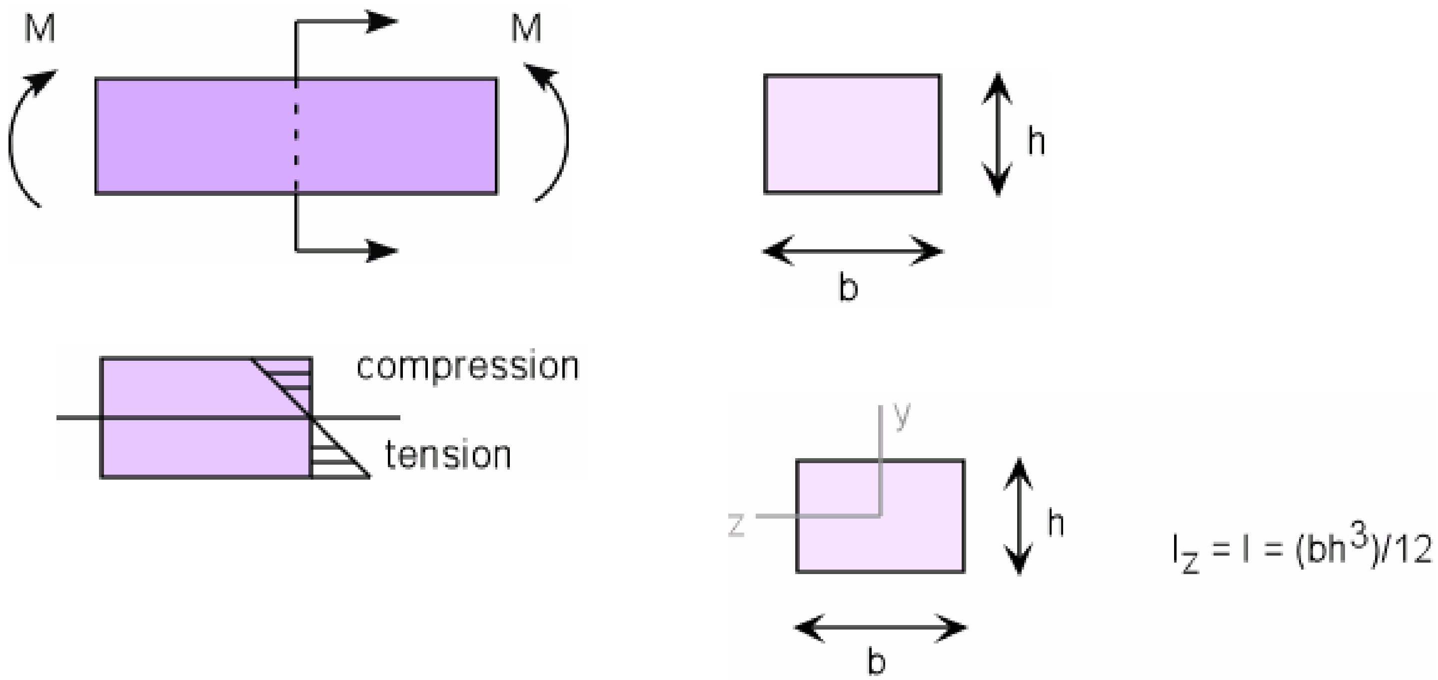
$$\varepsilon_y = \varepsilon_z = \frac{-\nu My}{EI}$$





Example

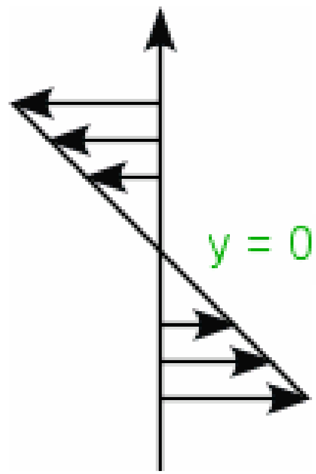
- Beam with rectangular cross-section



Beam in pure bending example, cont.

$$\sigma_x = My/I = \frac{M(h/2)}{bh^3/12}$$

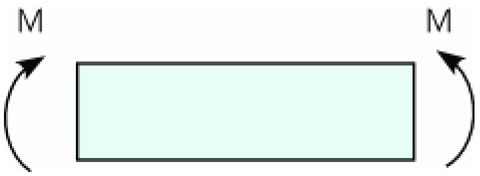
$$\sigma_x = \frac{-6M}{bh^2} \quad \text{(compression)}$$

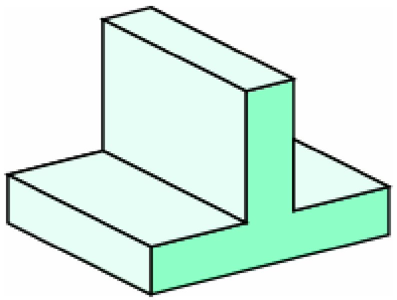
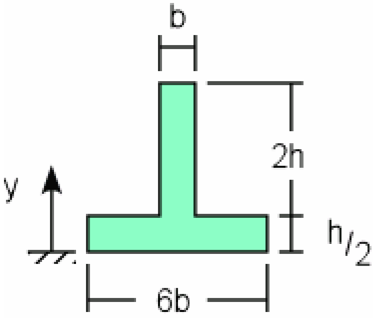


$$\sigma_x = \frac{6M}{bh^2} \quad \text{(tension)}$$

Example

- Find the maximum tensile and compressive stresses in the I-beam
- Beam in pure bending -- $\sigma = \dfrac{Mc}{I}$



Review of centroids

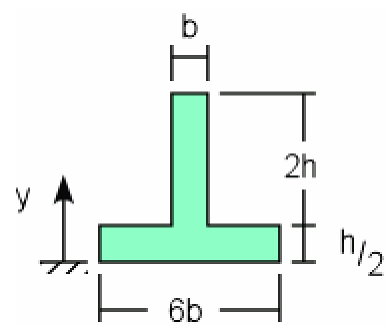
- Recall Parallel Axis Theorem

$$I = I_c + Ad^2$$

  - I is the moment of inertia about any point
  - I_c is moment of inertia about the centroid
  - A is area of section
  - d is distance from section centroid to axis of I

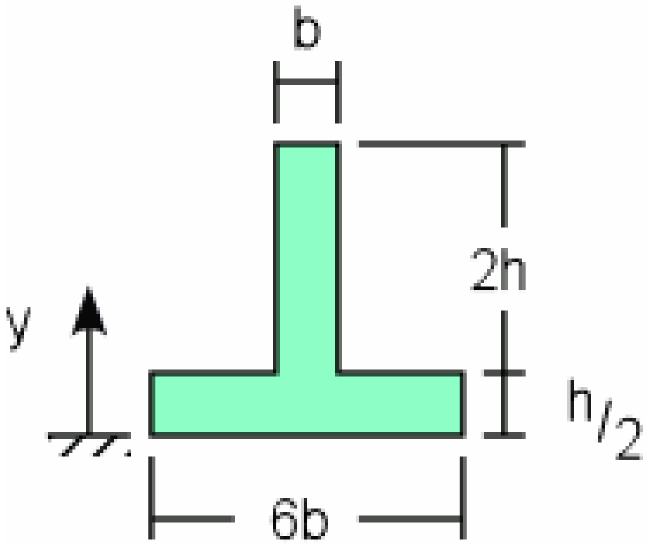
Example, cont.

- Recall:
  - c and I defined with respect to the neutral axis (NA)
- First, we'll need to find I
- So…
  - We need to find the neutral axis
  - Assume the I-beam is homogenous
  - NA passes through the centroid of the cross-section



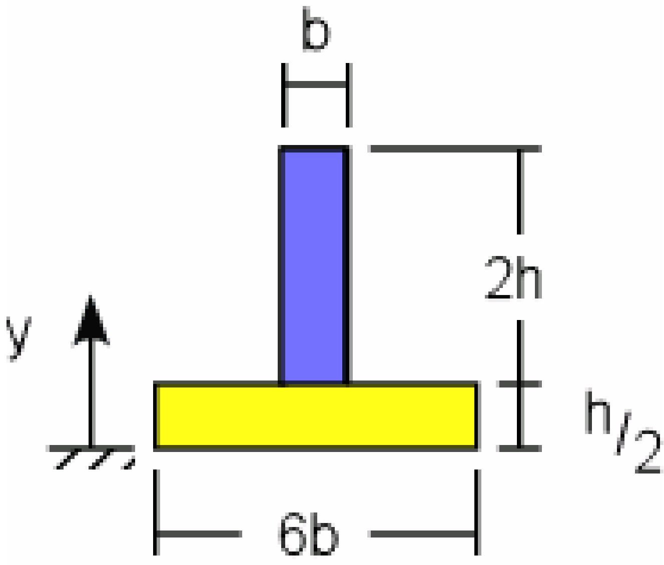
Finding the centroid

- Neutral axis passes through the centroid of the cross-sectional area
- Divide into simple-shaped sections to find the centroid of a composite area



$$\bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i}$$

$$\bar{y} = \frac{\left(\frac{h}{4}\right)(3hb) + \left(\frac{h}{2} + h\right)(2hb)}{3hb + 2hb} = \frac{3}{4}h$$

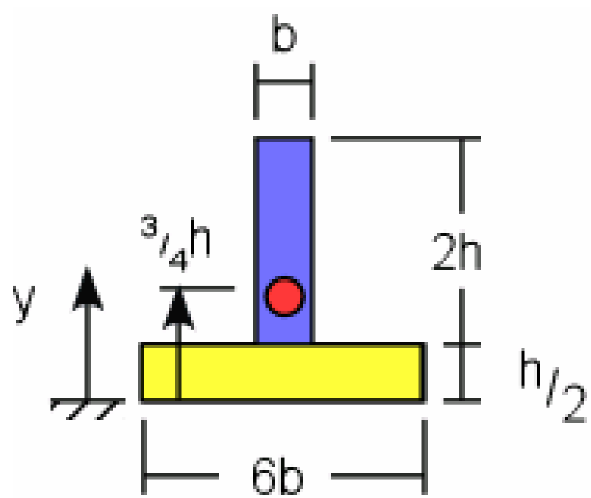


Finding I – Moment of Inertia

- Red dot shows centroid of the composite area that we just found
- Similarly, we can find the moment of inertia of a composite area

$$I_z = \sum I_{i_z}$$

$$I = I_{yellow} + I_{blue}$$



- Use parallel axis theorem to find I for the blue and yellow areas
  - Moment of inertia about a different point is the moment of inertia of the section about its centroid plus the area of the section times square of the distance to the point

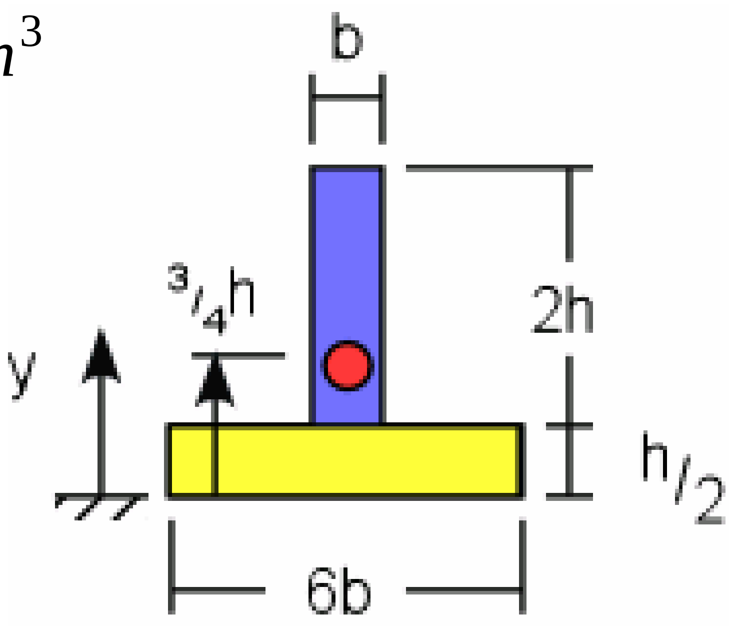
$$I = I_c + Ad^2$$

Finding I – Moment of Inertia, cont.

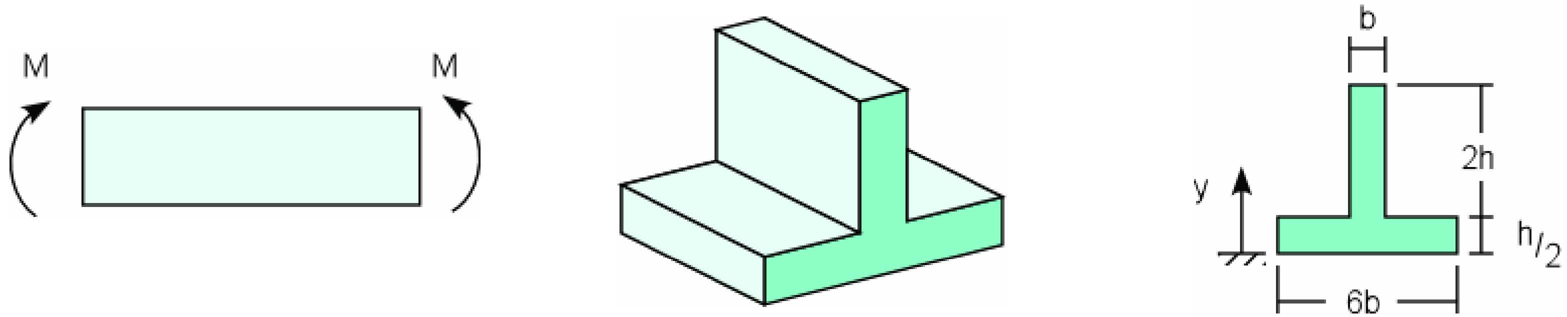
$$I_{yellow} = \frac{(6b)\left(\frac{h}{2}\right)^3}{12} + (3hb)\left(\frac{h}{2}\right)^2 = \frac{13}{16}bh^3$$

$$I_{blue} = \frac{(b)(2h)^3}{12} + (2hb)\left(\frac{3}{4}h\right)^2 = \frac{43}{24}bh^3$$

$$I = I_{blue} + I_{yellow} = \frac{125}{48}bh^3$$



Finding the bending stresses



- Maximum tensile stress occurs at the base

$$\sigma_{base} = \frac{Mc}{I} = \frac{M\left(\frac{3}{4}h\right)}{\frac{125}{48}bh^3} = \frac{36}{125}\left(\frac{M}{bh^2}\right)$$

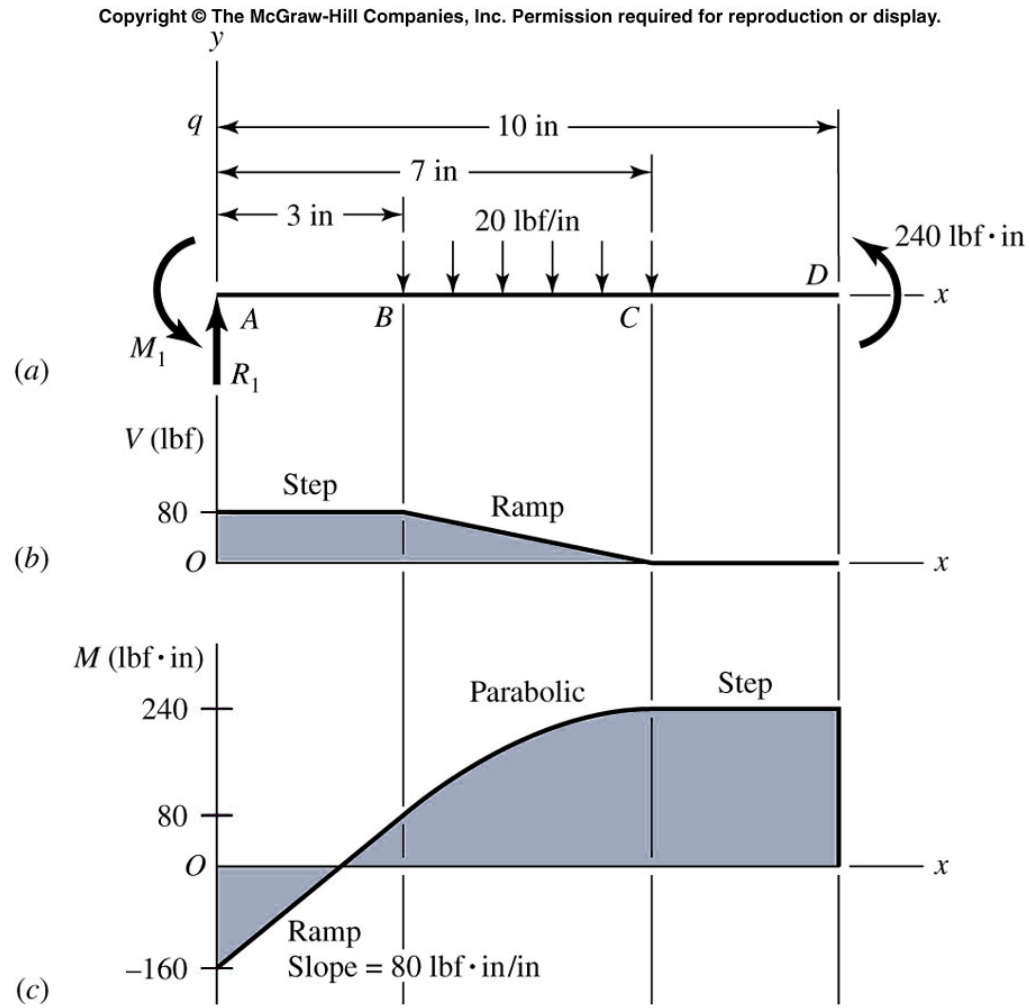
- Maximum compressive stress occurs at the top

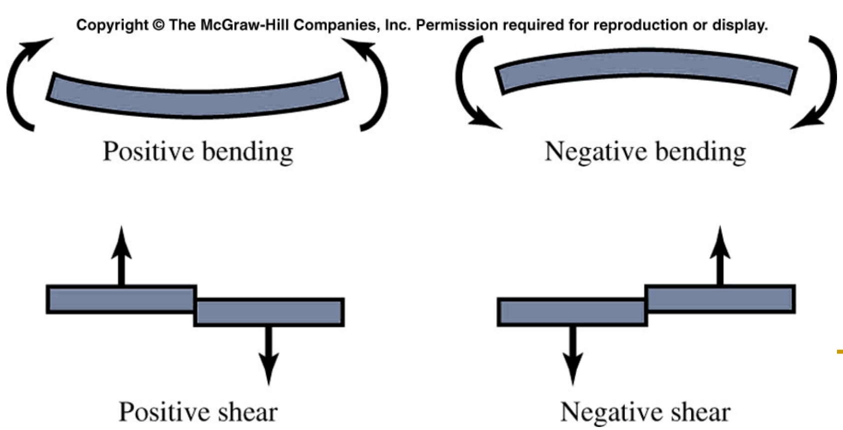
$$\sigma_{top} = \frac{Mc}{I} = \frac{-M\left(\frac{7}{4}h\right)}{\frac{125}{48}bh^3} = \frac{-84}{125}\left(\frac{M}{bh^2}\right)$$

- Note, y is the distance from the point of interest (top or base) to the neutral axis
  - Total height of cross-section = 2h + h/2 = 10h/4
  - Neutral axis is at 3h/4

Beams in bending and shear

- Assumptions for the analytical solution:
  - $\sigma_x = {}^{Mc}/_{I}$ holds even when moment is not constant along the length of the beam
  - τ_{xy} is constant across the width





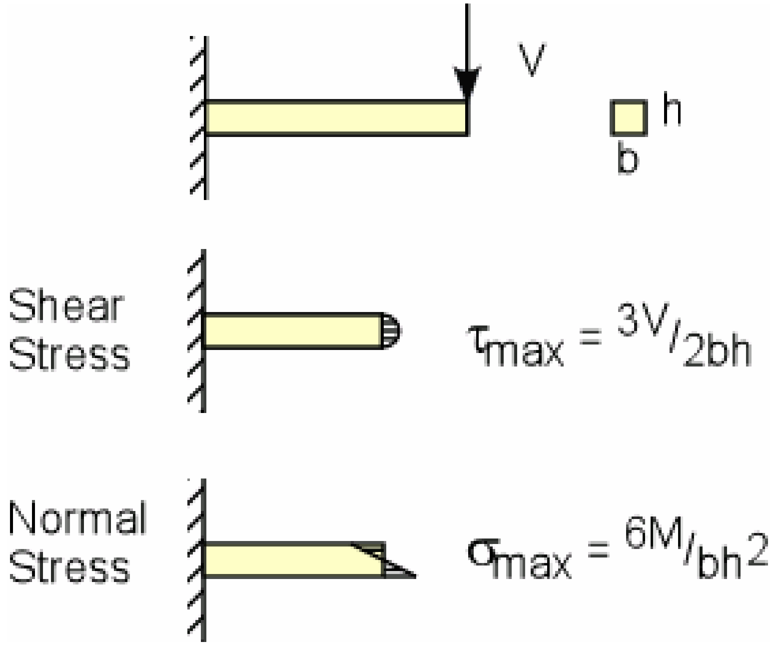
Calculating the shear stress for beams in bending

$$\tau_{xy} = \frac{VQ}{Ib}$$

- V(x) = shear force
- $I = I_z$ = area moment of inertia about NA (neutral axis)
- b(y) = width of beam
- $$Q(y) = \int_{A'} y\,dA'$$
  - Where A' is the area between y=y' and the top (or bottom) of the beam cross-section
  - General observations about Q:
    - Q is 0 at the top and bottom of the beam
    - Q is maximum at the neutral axis
    - τ = 0 at top and bottom of cross-section
    - τ = max at neutral axis
- Note, V and b can be functions of y

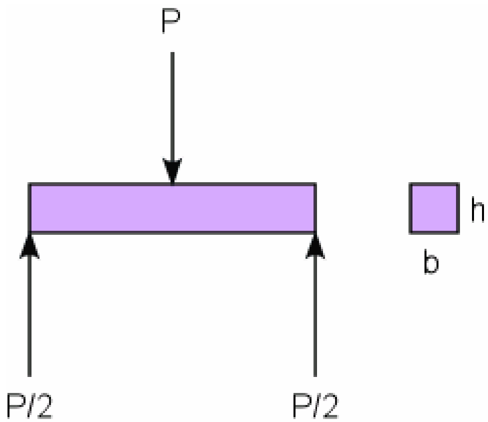
Usually we have common cross-sections

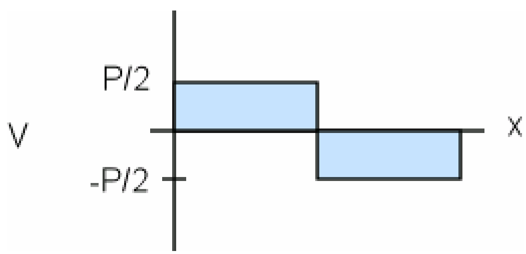
- τ_{max} for common shapes on page 136
- Example:
  - Rectangular cross-section
  - $$Q = \frac{b}{2}\left[\left(\frac{h}{2}\right)^2 - y^2\right]$$
  - Shear and normal stress distributions across the cross-section

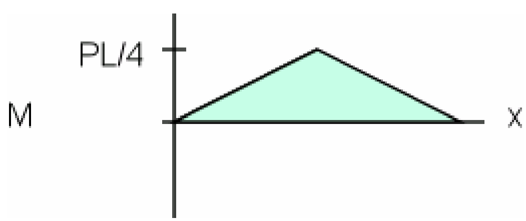


Relative magnitudes of normal and shear stresses

Rectangular cross-section







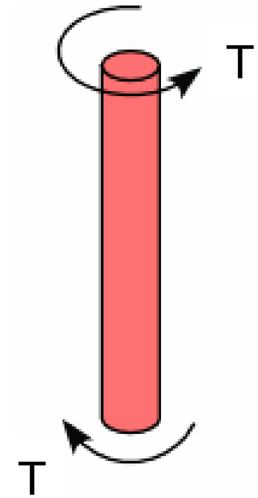
$$\tau_{\max} = \frac{3}{4}\frac{V}{A} = \frac{3}{4}\left(\frac{\frac{P}{2}}{bh}\right) = \frac{3P}{8bh}$$

$$\sigma_{\max} = \frac{My}{I} = \frac{3}{2}\left(\frac{PL}{bh^2}\right)$$

$$\frac{\tau_{\max}}{\sigma_{\max}} = \frac{1}{4}\left(\frac{h}{l}\right)$$

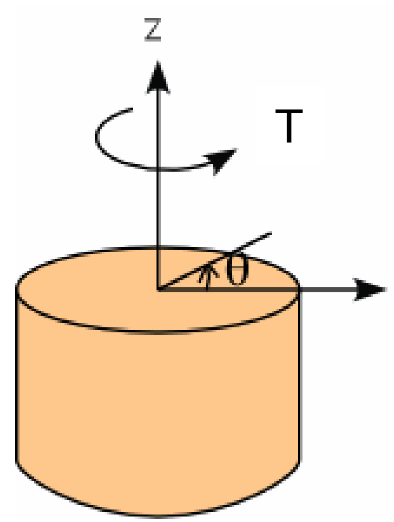
For THIS loading, if h << L, then
τ_{max} << σ_{max} and τ can be neglected

Shafts in torsion



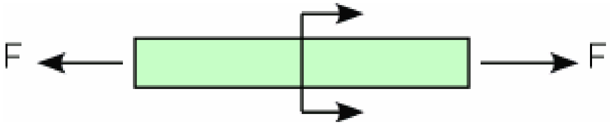
- Assumptions
  - Constant moment along length
  - No lengthening or shortening of shaft
  - Linearly elastic
  - Homogenous

$$\tau_{z\theta} = \frac{Tr}{J}$$



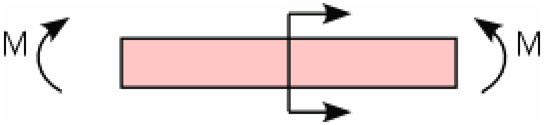
  - Where J is the polar moment of inertia $J = \int_A r^2 dA$
  - Note:
    - Circular shaft $J = \frac{\pi d^4}{32}$
    - Hollow shaft $J = \frac{\pi}{32}\left[d_o^{\,4} - d_i^{\,4}\right]$

Recap: Primary forms of loading



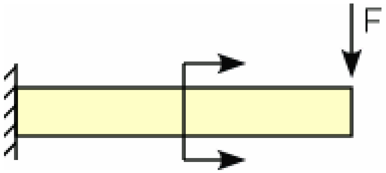
- Axial

$$\sigma = \frac{F}{A}$$



- Pure bending

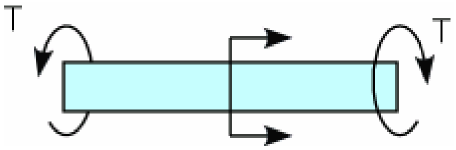
$$\sigma = \frac{Mc}{I}$$



- Bending and shear

$$\sigma = \frac{Mc}{I}$$

$$\tau = \frac{VQ}{Ib}$$



- Torsion

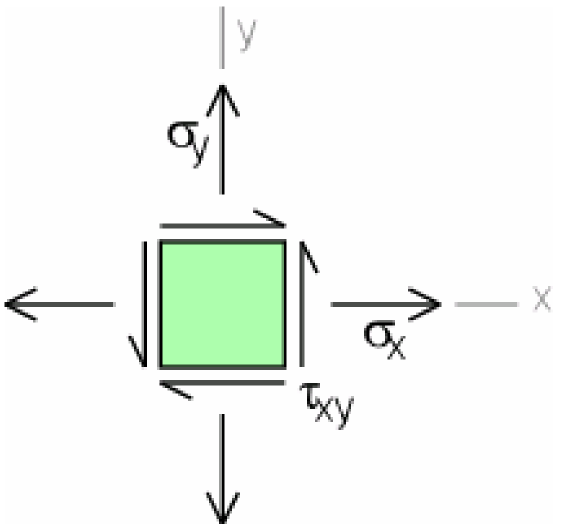
$$\tau = \frac{Tr}{J}$$

Questions

- So, when I load a beam in pure bending, is there any shear stress in the material? What about uniaxial tension?
- Yes, there is!
- The equations on the previous slide don't tell the whole story
- Recall:
  - When we derived the equations above, we always sliced the beam (or shaft) perpendicular to the long axis
- If we make some other cut, we will in general get a different stress state

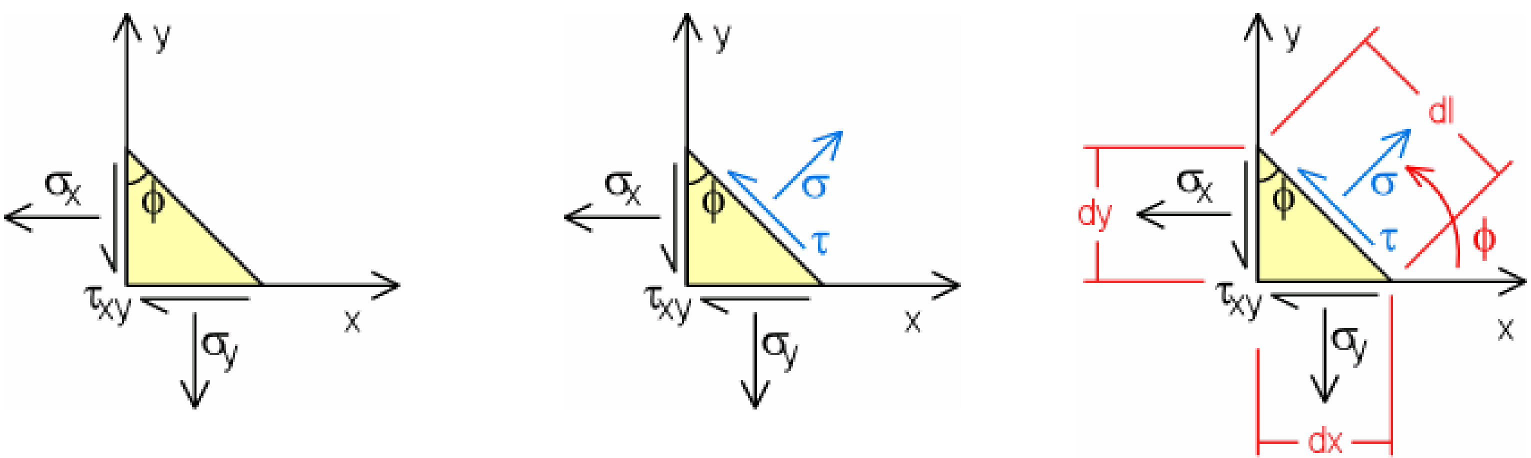
General case of planar stress

- Infinitesimal piece of material:
- A general state of planar stress is called a biaxial stress state
- Three components of stress are necessary to specify the stress at any point
  - σ_x
  - σ_y
  - τ_{xy}

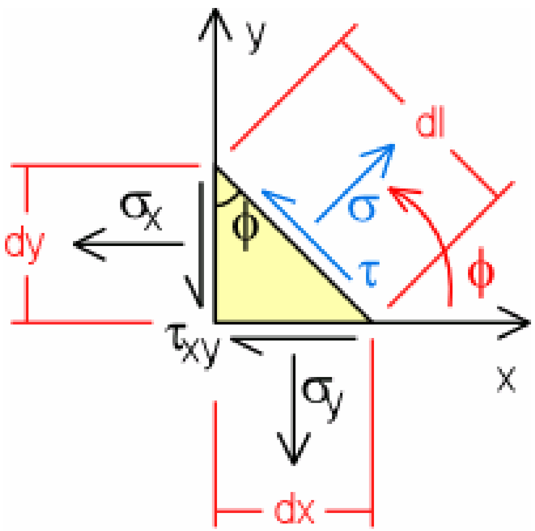


Changing orientation

- Now let's slice this element at any arbitrary angle to look at how the stress components vary with orientation
- We can define a normal stress (σ) and shear stress (τ)
- Adding in some dimensions, we can now solve a static equilibrium problem…



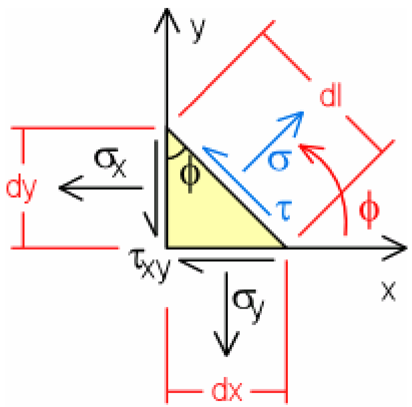
Static equilibrium equations



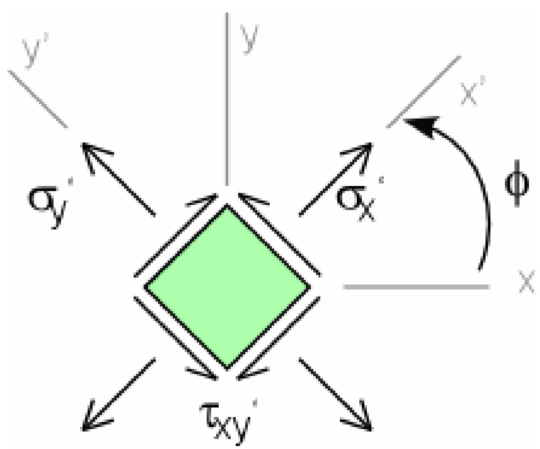
$$y = l\cos\phi$$
$$x = l\sin\phi$$

$$\sigma(\cos\phi)dl - \tau(\sin\phi)dl = -\sigma_x dy - \tau_{xy} dx$$
$$\sigma(\sin\phi)dl + \tau(\cos\phi)dl = -\sigma_y dx - \tau_{xy} dy$$

From equilibrium…



- We can find the stresses at any arbitrary orientation (σ_x', σ_y', τ_{xy}')



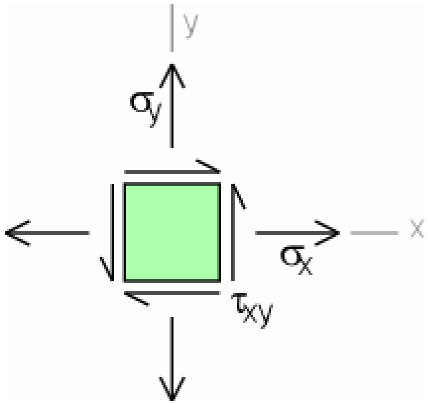
$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos(2\phi) + \tau_{xy}\sin(2\phi)$$

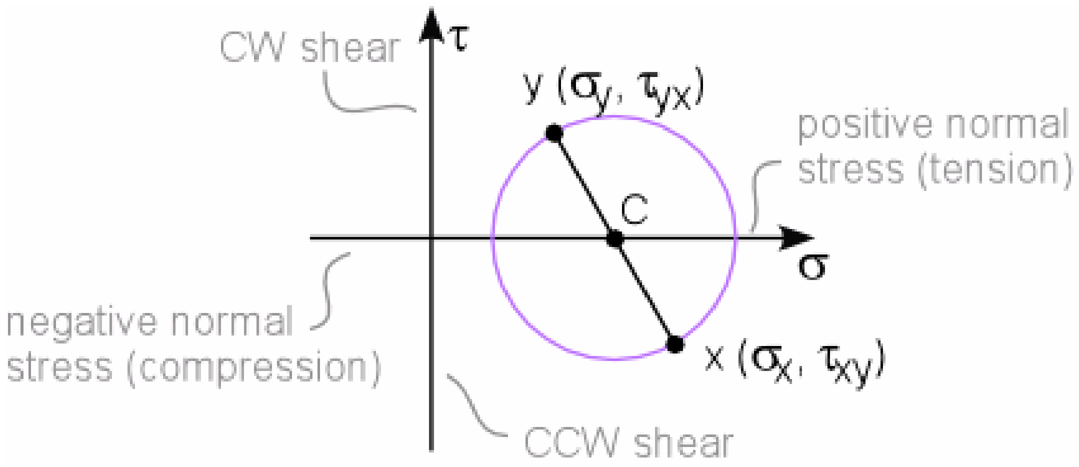
$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos(2\phi) - \tau_{xy}\sin(2\phi)$$

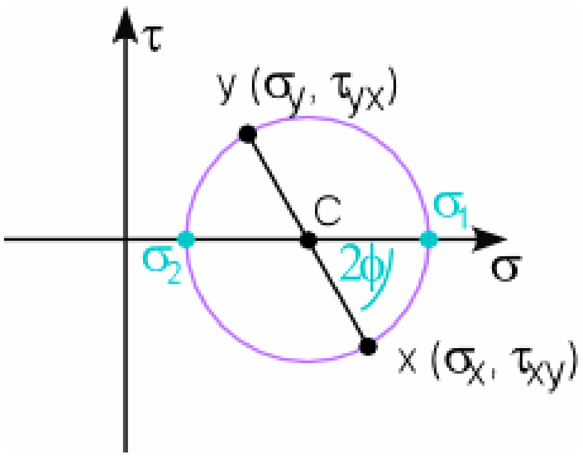
$$\tau_{xy}' = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin(2\phi) + \tau_{xy}\cos(2\phi)$$

Mohr's Circle

- These equations can be represented geometrically by Mohr's Circle
- Stress state in a known orientation:
- Draw Mohr's circle for stress state:
- ϕ is our orientation angle, which can be found by measuring FROM the line XY to the orientation axis we are interested in

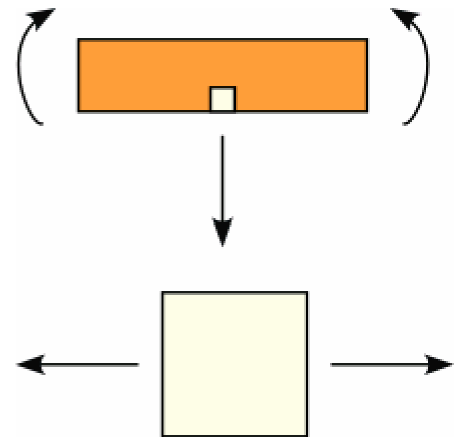




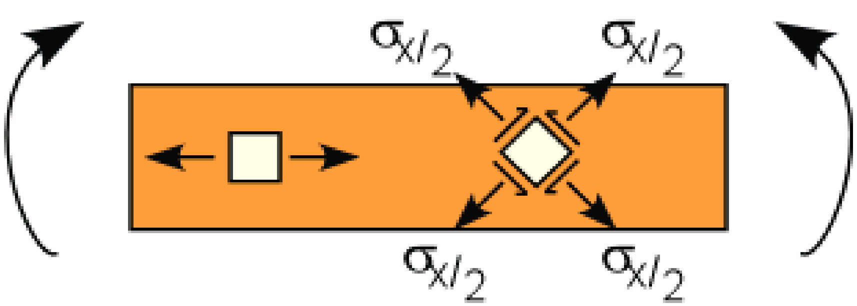


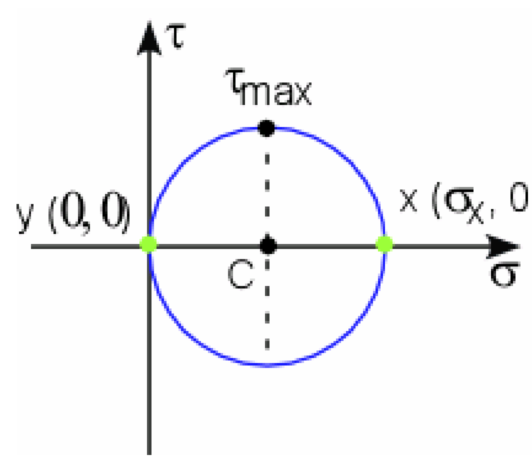
Question from before…

- Is a beam in pure bending subjected to any shear stress?
- Take an element…
- Draw Mohr's Circle
- τ_{max} occurs at the orientation $2\phi = 90^{o}$
  - $\phi = 45^{o}$



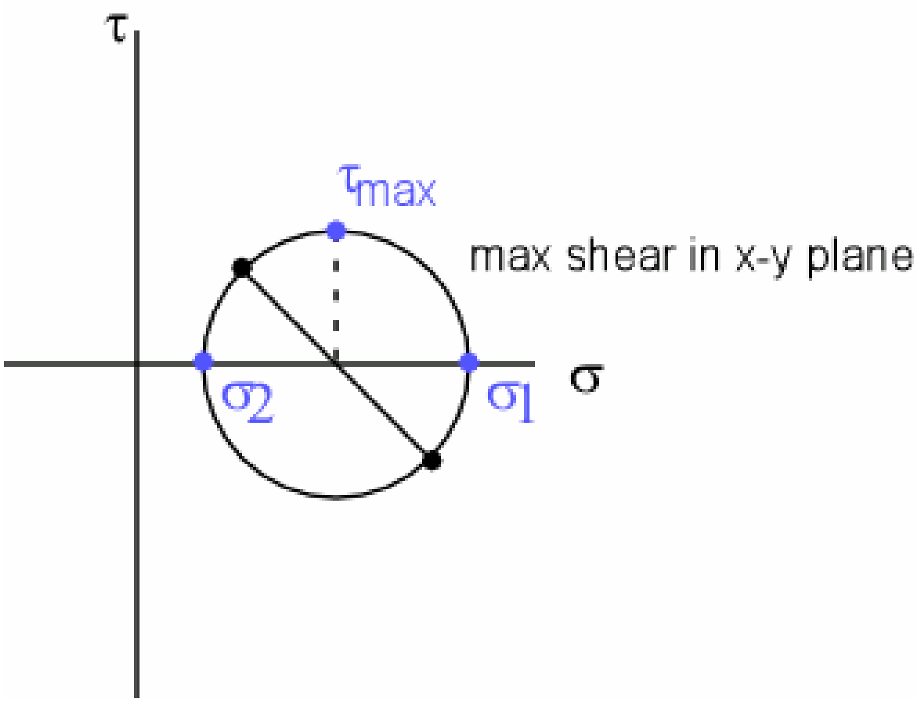
$$\sigma = \frac{My}{I}$$





Special points on Mohr's Circle

- $\sigma_{1,2}$ – Principal stresses
  - At this orientation, one normal stress is maximum and shear is zero
  - Note, $\sigma_1 > \sigma_2$
- τ_{max} – Maximum shear stress (in plane)
  - At this orientation, normal stresses are equal and shear is at a maximum
- Why are we interested in Mohr's Circle?

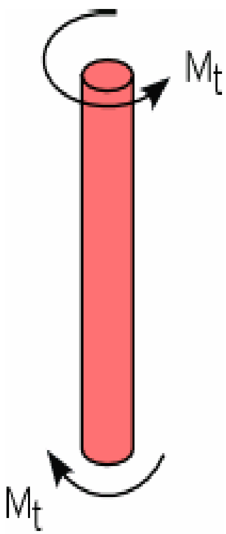


Mohr's Circle, cont.

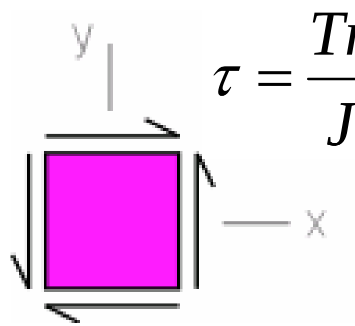
- A shaft in torsion has a shear stress distribution:

$$\tau = \frac{Tr}{J}$$

- Why does chalk break like this…?

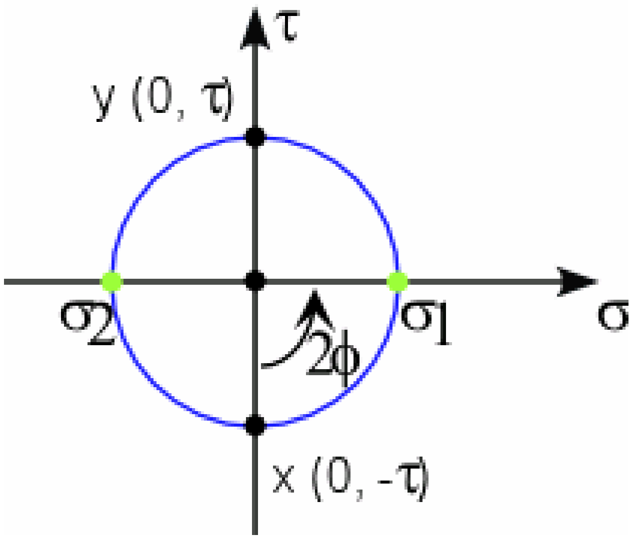


Look at an element and its stress state:



$$\tau = \frac{Tr}{J}$$

Mohr's circle for our element:



- σ_1 and σ_2 are at $2\phi = 90^o$
- Therefore $\phi = 45\ ^o$
- This is the angle of maximum shear!
  - The angle of maximum shear indicates how the chalk will fail in torsion

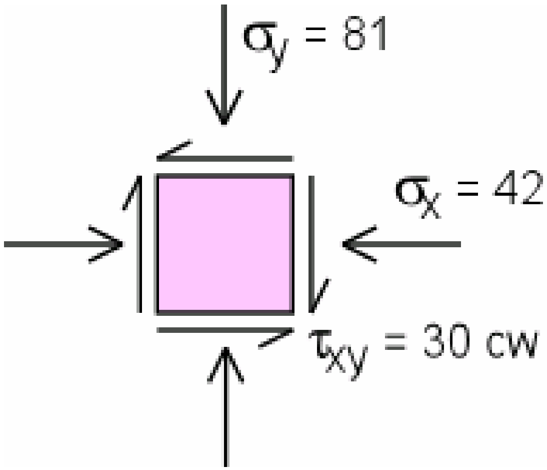
Example #1

- $\sigma_x = -42$
- $\sigma_y = -81$
- $\tau_{xy} = 30$ cw
- x at (σ_x, τ_{xy})
  - x at (-42, 30)
- y at (σ_y, τ_{yx})
  - y at (-81, -30)
- Center

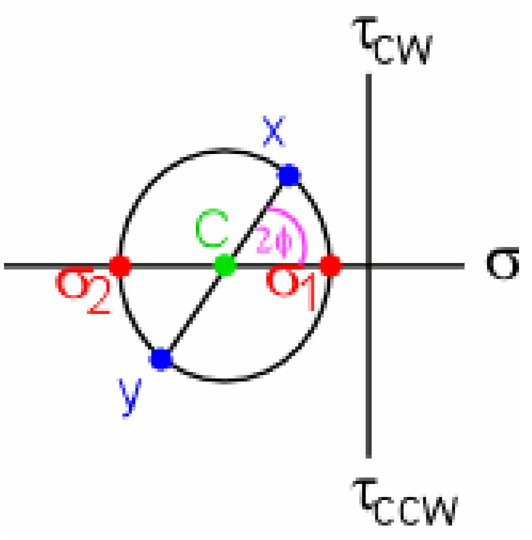
$$\left(\frac{\sigma_x + \sigma_y}{2}, 0\right) = \left(\frac{-42 - 81}{2}, 0\right) = (-61.5, 0)$$

- Radius

$$R = \sqrt{\tau_{xy}{}^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} = \sqrt{30^2 + \left(\frac{-42 - (-81)}{2}\right)^2} = 35.8$$

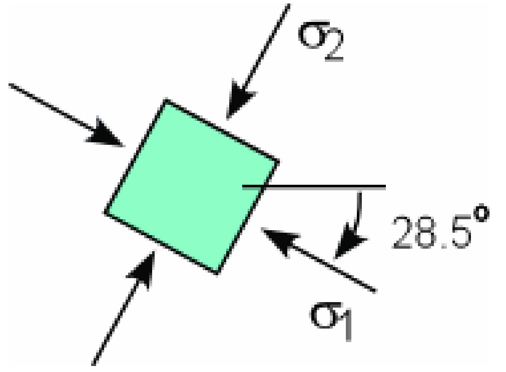


Example #1, cont.



- Now we have:
  - x at (-42, 30)
  - y at (-81, -30)
  - C at (-61.5, 0)
  - R = 35.8
- Find principal stresses:
  - $\sigma_1 = C_x + R = -25.7$
  - $\sigma_2 = C_x - R = -97.3$
  - $\tau_{max} = R = 35.8$
  - Orientation:

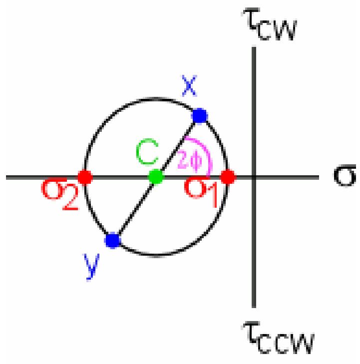
$$2\phi = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \tan^{-1}\left(\frac{60}{39}\right) = 56.9^o\, cw$$

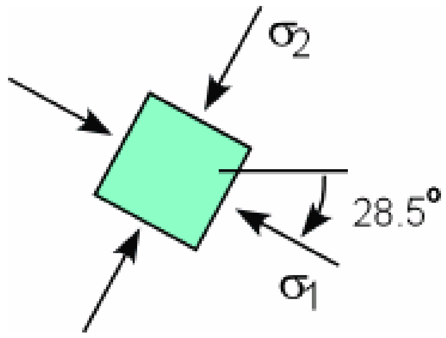


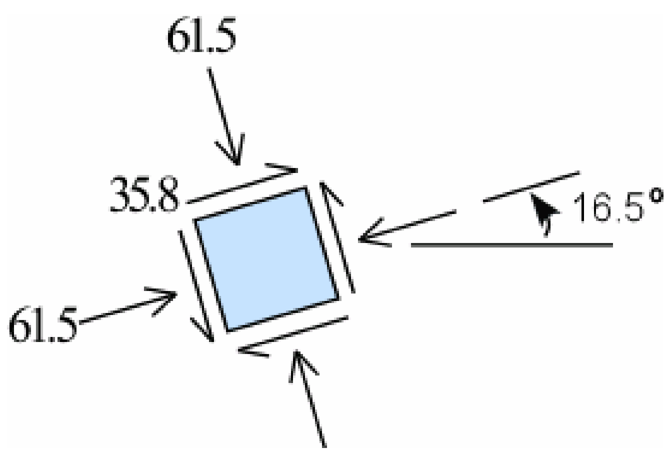
Recall, 2ϕ is measured from the line XY to the principal axis. This is the same rotation direction you use to draw the PRINCIPAL ORIENTATION ELEMENT

Example #1, cont.

- Orientation of maximum shear
- At what orientation is our element when we have the case of max shear?
- From before, we have:
  - $\sigma_1 = C_x + R = -25.7$
  - $\sigma_2 = C_x - R = -97.3$
  - $\tau_{max} = R = 35.8$
  - $\phi = 28.5\,^o$ CW
- $\phi_{max} = \phi_{1,2} + 45^o$ CCW
- $\phi_{max} = 28.5\,^o$ CW $+ 45^o$ CCW
  - $16.5\,^o$ CCW







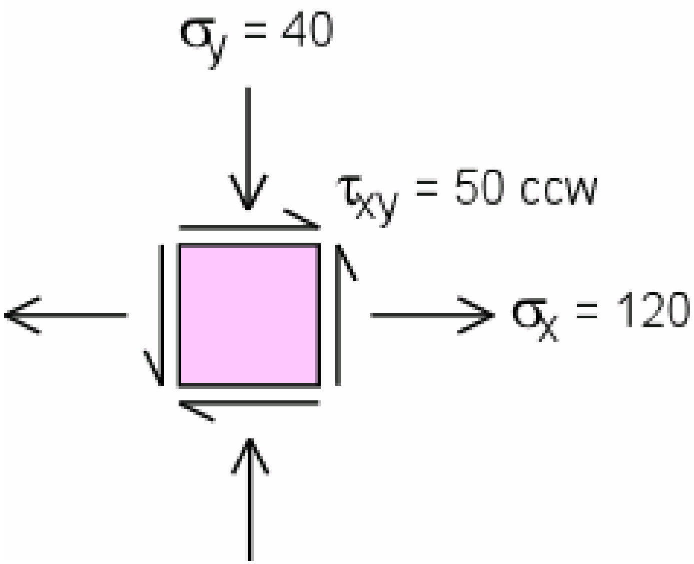
Example #2

- $\sigma_x = 120$
- $\sigma_y = -40$
- $\tau_{xy} = 50$ ccw
- x at (σ_x, τ_{xy})
  - x at (120, -50)
- y at (σ_y, τ_{yx})
  - y at (-40, 50)
- Center

$$\left(\frac{\sigma_x + \sigma_y}{2}, 0\right) = \left(\frac{120 - 40}{2}, 0\right) = (40, 0)$$

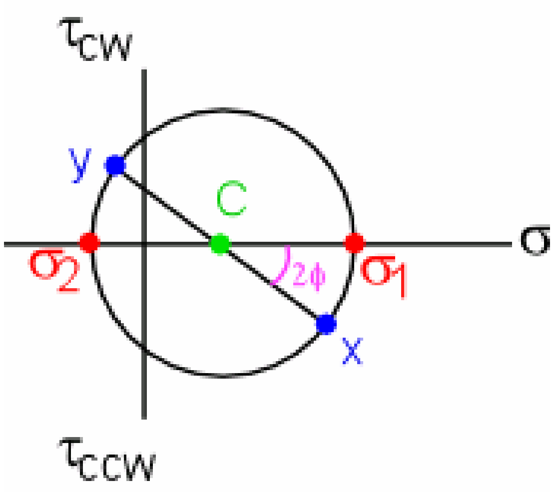
- Radius

$$R = \sqrt{\tau_{xy}^{\;2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2} = \sqrt{50^2 + \left(\frac{120 - (-40)}{2}\right)^2} = 94.3$$



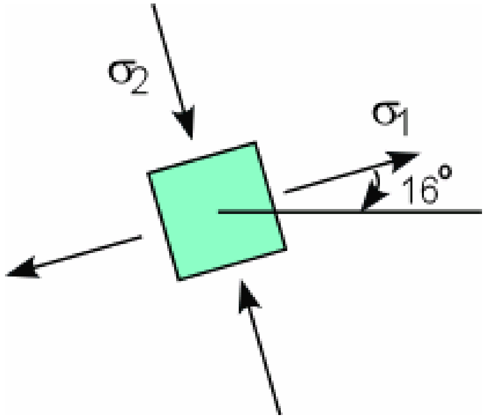
Example #2, cont.

- Now we have:
  - x at (120, -50)
  - y at (-40, 50)
  - C at (40, 0)
  - R = 94.3



- Find principal stresses:
  - $\sigma_1 = C_x + R = 134.3$
  - $\sigma_2 = C_x - R = -54.3$
  - $\tau_{max} = R = 94.3$
  - Orientation:

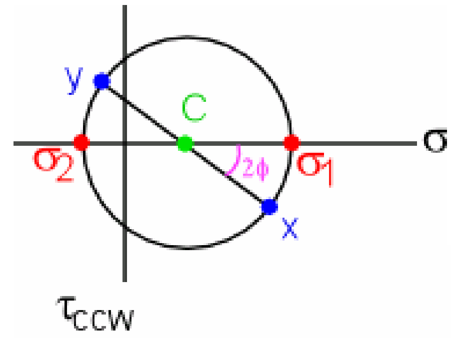
$$2\phi = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \tan^{-1}\left(\frac{2(-50)}{120-(-40)}\right) = 32.0^o\,ccw$$

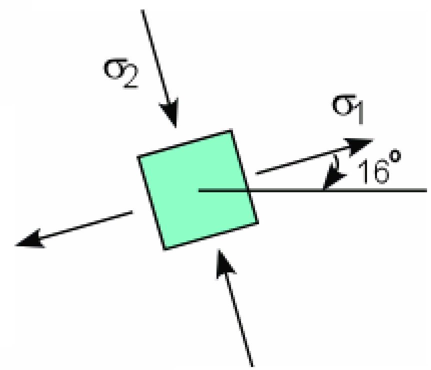


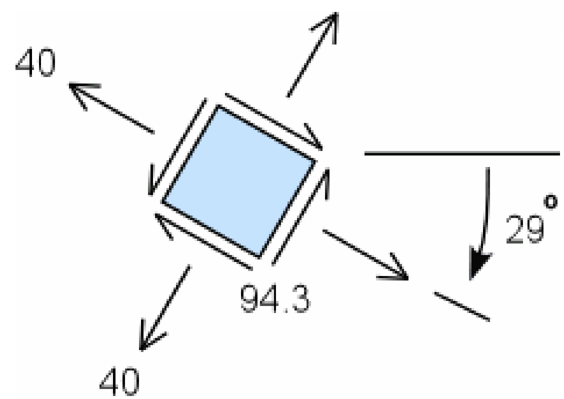
Recall, 2ϕ is measured from the line XY to the principal axis. This is the same rotation direction you use to draw the PRINCIPAL ORIENTATION ELEMENT

Example #2, cont.

- Orientation of maximum shear
- At what orientation is our element when we have the case of max shear?
- From before, we have:
  - $\sigma_1 = C_x + R = 134.3$
  - $\sigma_2 = C_x - R = -54.3$
  - $\tau_{max} = R = 94.3$
  - $\phi = 16.0\ ^o$ CCW
- $\phi_{max} = \phi_{1,2} + 45^o$ CCW
- $\phi_{max} = 16.0\ ^o$ CCW $+ 45^o$ CCW
  - $61.0\ ^o$ CCW $= 90.0 - 61.0\ ^o$ CW
  - $= 29.0\ ^o$ CW

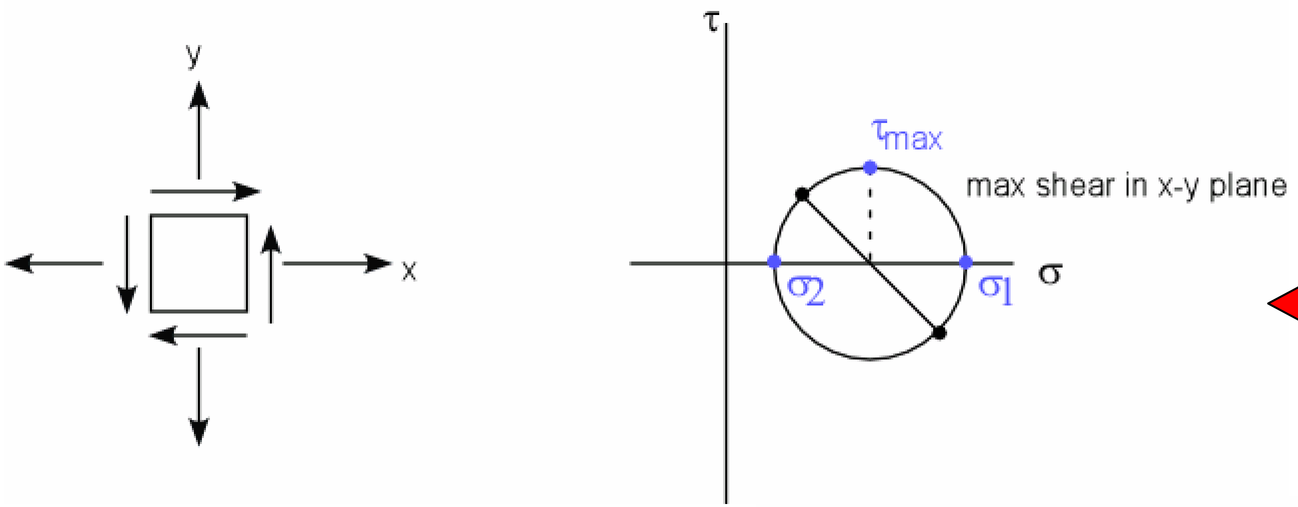






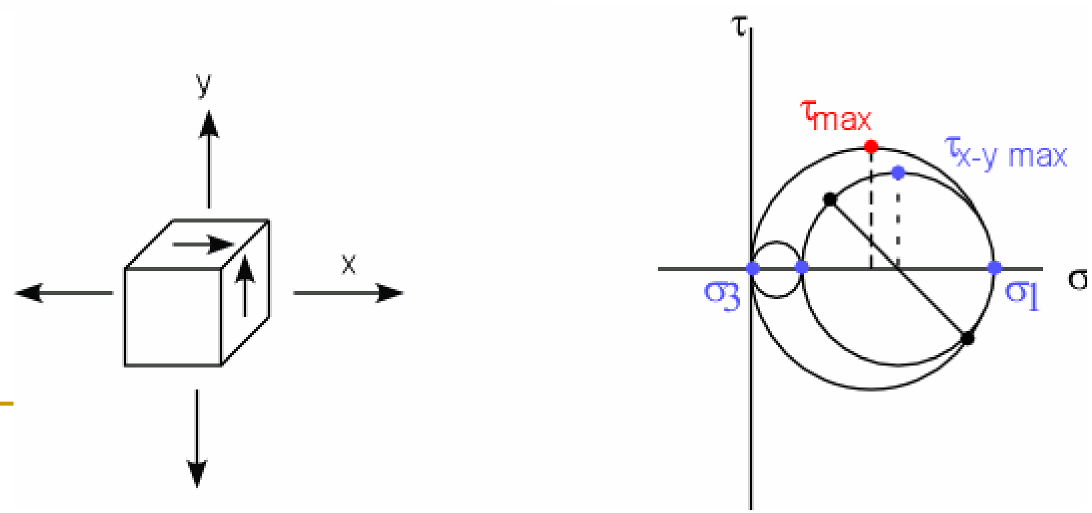
3-D Mohr's Circle and Max Shear

- Max shear in a plane vs. Absolute Max shear



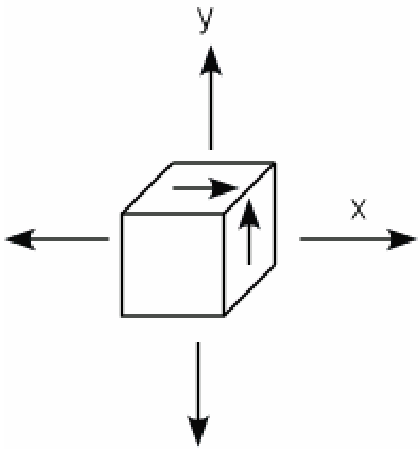
Biaxial State of Stress

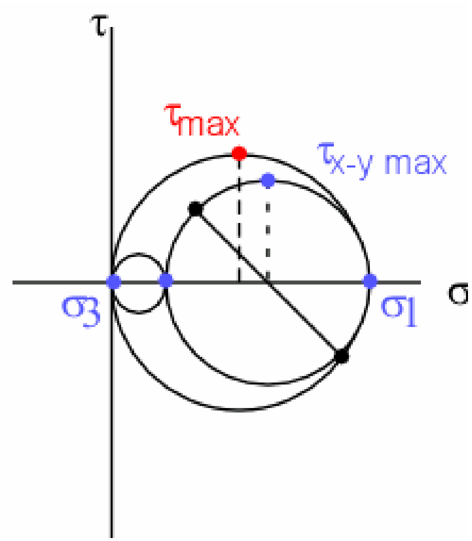
Still biaxial, but consider the 3-D element



3-D Mohr's Circle

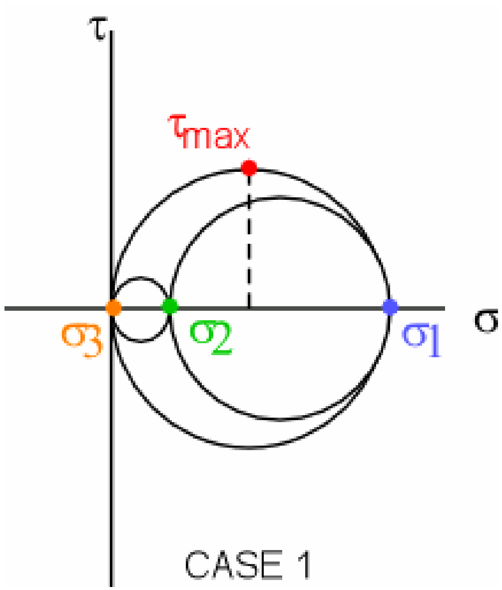
- τ_{max} is oriented in a plane 45º from the x-y plane
  - (2ϕ = 90º)
- When using "max shear", you must consider τ_{max}
  - (Not $\tau_{x\text{-}y\ max}$)

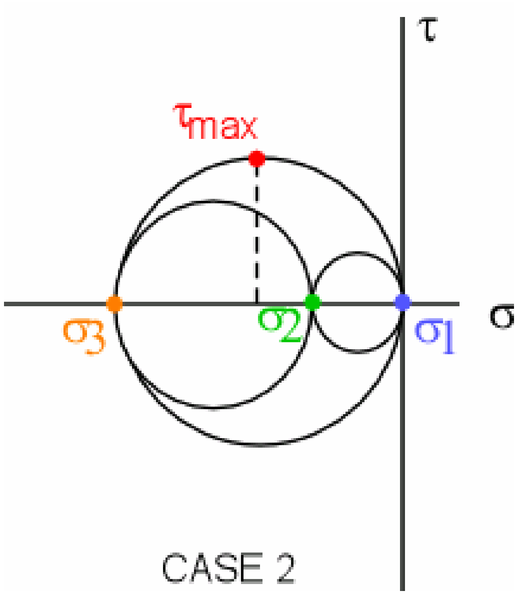


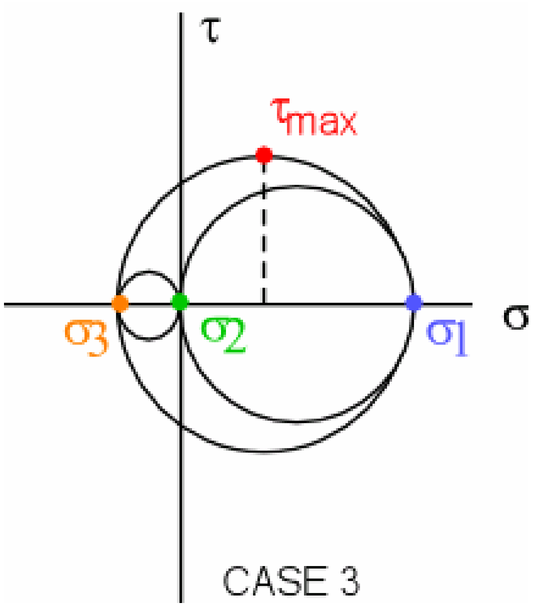


Out of Plane Maximum Shear for Biaxial State of Stress

- Case 1
  - $\sigma_{1,2} > 0$
  - $\sigma_3 = 0$
  - $\tau_{max} = \dfrac{\sigma_1}{2}$
- Case 2
  - $\sigma_{2,3} < 0$
  - $\sigma_1 = 0$
  - $\tau_{max} = \dfrac{|\sigma_3|}{2}$
- Case 3
  - $\sigma_1 > 0, \sigma_3 < 0$
  - $\sigma_2 = 0$
  - $\tau_{max} = \dfrac{|\sigma_1 - \sigma_3|}{2}$







Additional topics we will cover

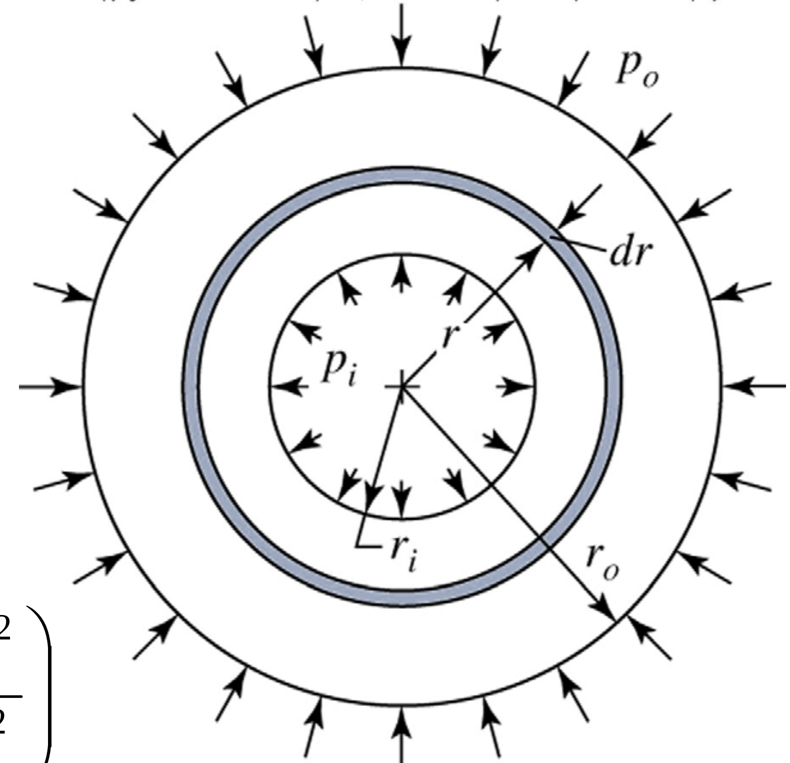
- 3-13 stress concentration
- 3-14 pressurized cylinders
- 3-18 curved beams in bending
- 3-19 contact stresses

Stress concentrations

- We had assumed no geometric irregularities
- Shoulders, holes, etc are called discontinuities
  - Will cause stress raisers
  - Region where they occur – stress concentration
- Usually ignore them for ductile materials in static loading
  - Plastic strain in the region of the stress is localized
  - Usually has a strengthening effect
- Must consider them for brittle materials in static loading
  - Multiply nominal stress (theoretical stress without SC) by K_t, the stress concentration factor.
  - Find them for variety of geometries in Tables A-15 and A-16
- We will revisit SC's…

Stresses in pressurized cylinders

- Pressure vessels, hydraulic cylinders, gun barrels, pipes
- Develop radial and tangential stresses
  - Dependent on radius



$$\sigma_t = \frac{r_i^2 p_i}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$

$$\sigma_r = \frac{r_i^2 p_i}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$

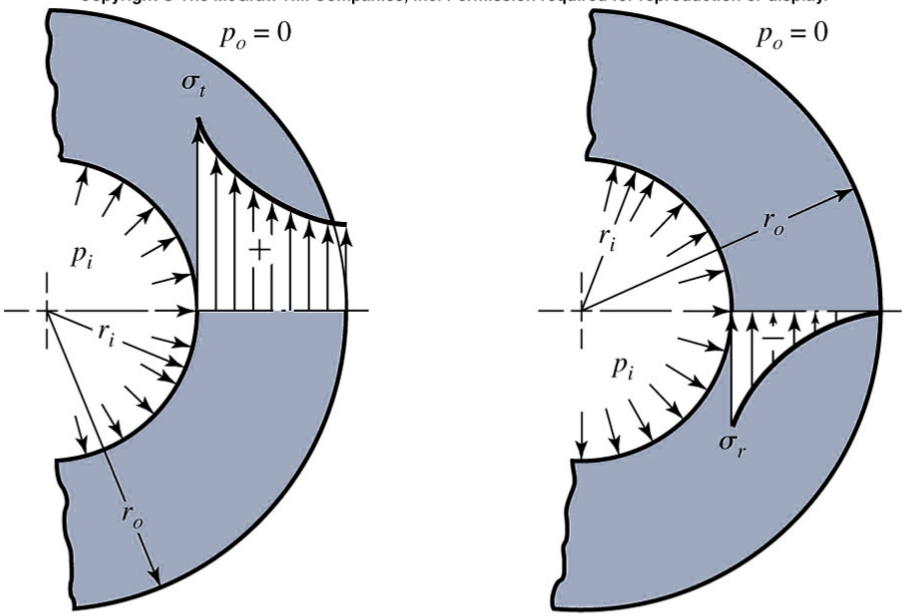
for $p_o = 0$

Stresses in pressurized cylinders, cont.

- Longtudincal stresses exist when the end reactions to the internal pressure are taken by the pressure vessel itself

$$\sigma_l = \frac{r_i^2 p_i}{r_o^2 - r_i^2}$$

- These equations only apply to sections taken a significant distance from the ends and away from any SCs



(*a*) Tangential stress distribution

(*b*) Radial stress distribution

Thin-walled vessels

- If wall thickness is 1/20th or less of its radius, the radial stress is quite small compared to tangential stress

$$(\sigma_t)_{avg} = \frac{pd_i}{2t}$$

$$(\sigma_t)_{\max} = \frac{p(d_i + t)}{2t}$$

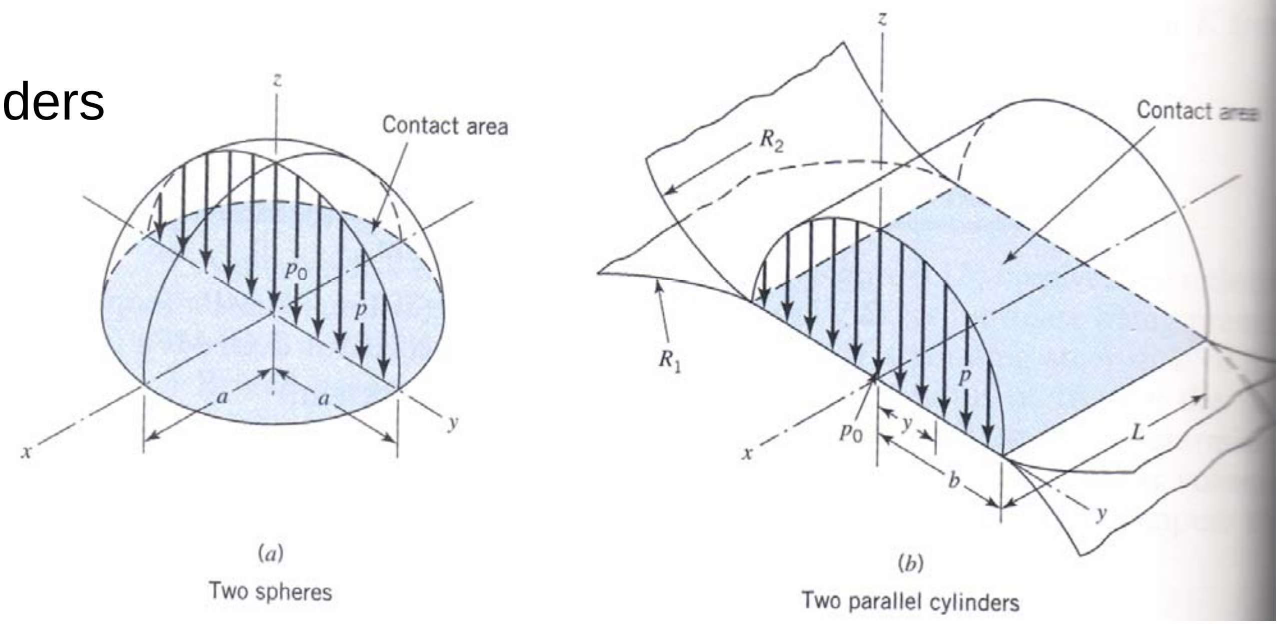
$$\sigma_l = \frac{pd_i}{4t}$$

Curved-surface contact stresses

- Theoretically, contact between curved surfaces is a point or a line
- When curved *elastic* bodies are pressed together, finite contact areas arise
  - Due to deflections
  - Areas tend to be small
  - Corresponding compressive stresses tend to be very high
  - Applied cyclically
    - Ball bearings
    - Roller bearings
    - Gears
    - Cams and followers
    - Result – fatigue failures caused by minute cracks
      - “surface fatigue”

Contact stresses

- Contact between spheres
  - Area is circular
- Contact between cylinders (parallel)
  - Area is rectangular
- Define maximum contact pressure (p_0)
  - Exists on the load axis
- Define area of contact
  - *a* for spheres
  - *b* and *L* for cylinders



Contact stresses - equations

- First, introduce quantity Δ, a function of Young's modulus (E) and Poisson's ratio (ν) for the contacting bodies

$$\Delta = \frac{1-\nu_1{}^2}{E_1} + \frac{1-\nu_2{}^2}{E_2}$$

- Then, for two spheres,

$$p_0 = 0.578\left(\sqrt[3]{\frac{F\left(1/R_1 + 1/R_2\right)}{\Delta^2}}\right) \qquad a = 0.908\left(\sqrt[3]{\frac{F\Delta}{1/R_1 + 1/R_2}}\right)$$

- For two parallel cylinders,

$$p_0 = 0.564\left(\sqrt{\frac{F\left(1/R_1 + 1/R_2\right)}{L\Delta}}\right) \qquad b = 1.13\left(\sqrt{\frac{F\Delta}{L\left(1/R_1 + 1/R_2\right)}}\right)$$

Contact stresses

- Contact pressure (p_0) is also the value of the surface compressive stress (σ_z) at the load axis
- Original analysis of elastic contact
  - 1881
  - Heinrich Hertz of Germany
- Stresses at the mating surfaces of curved bodies in compression:
  - Hertz contact stresses

Contact stresses

- Assumptions for those equations
  - Contact is frictionless
  - Contacting bodies are
    - Elastic
    - Isotropic
    - Homogenous
    - Smooth
  - Radii of curvature R_1 and R_2 are very large in comparison with the dimensions of the boundary of the contact surface

Elastic stresses below the surface along load axis (Figures4-43 and 4-45 in JMB)

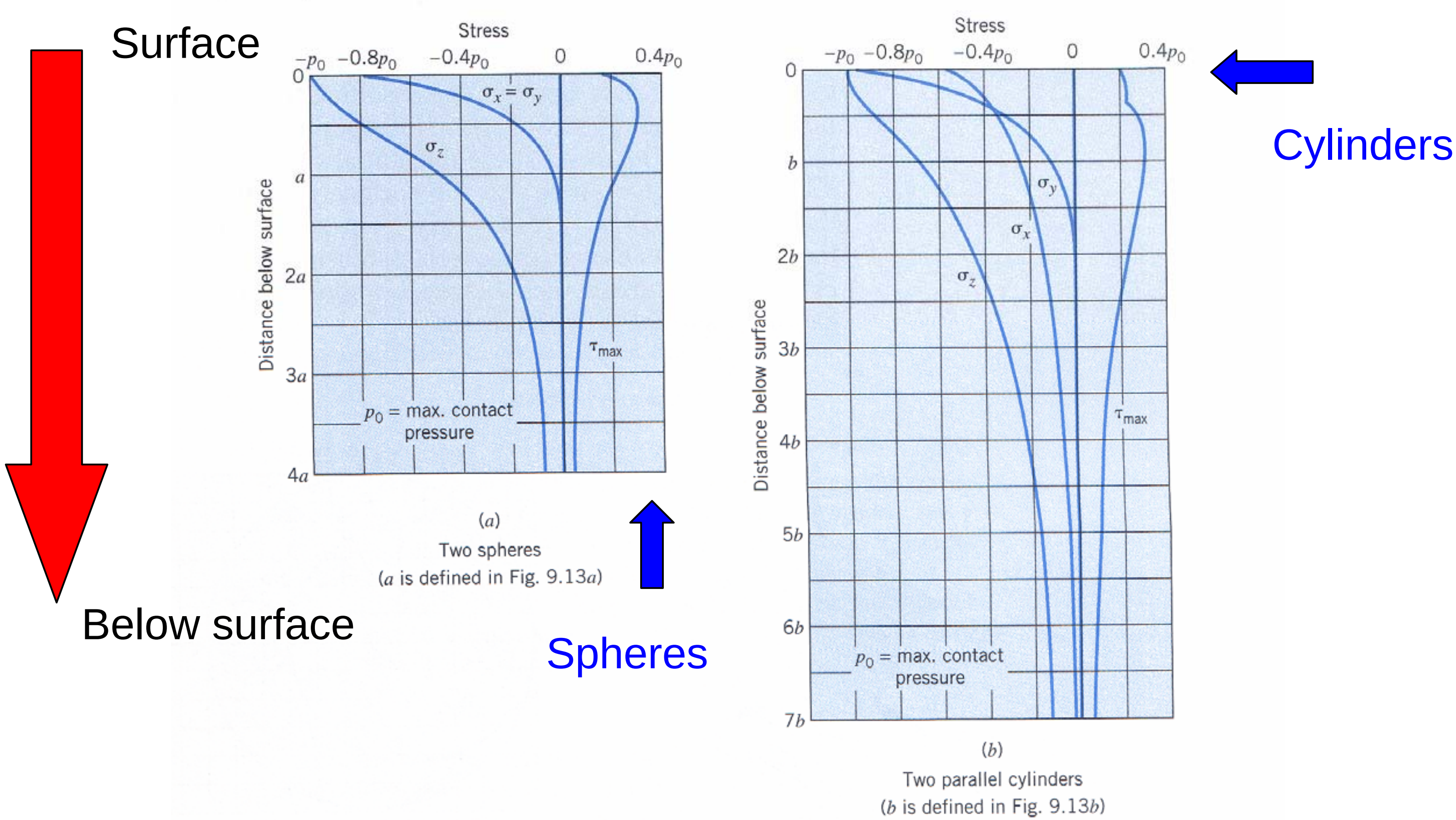
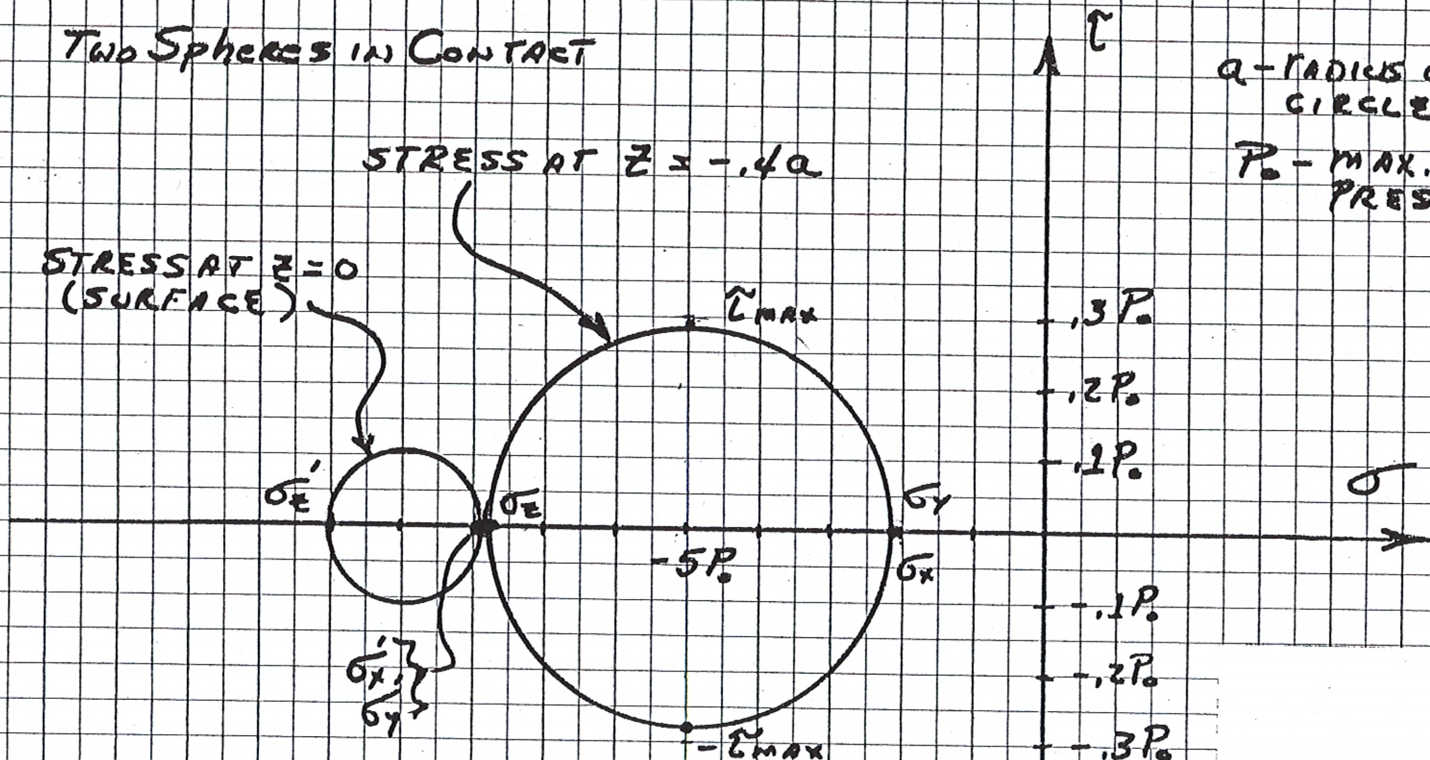


FIGURE 9.15
Elastic stresses below the surface, along the load axis (the z-axis; $x = 0$, $y = 0$; for $\nu = 0.3$).

MOHR'S CIRCLE

TWO SPHERES IN CONTACT



SURFACE (Z=0)

$\sigma_x' = \sigma_y' = -0.8P_o$

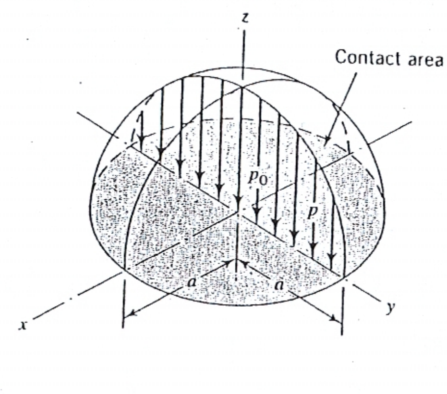
$\sigma_z' = -P_o$

AT Z = -0.4a

$\sigma_x = \sigma_y = -0.22P_o$

$\sigma_z = -0.82P_o$

$\tau_{max} = \pm 0.3P_o$



(a)

Two spheres

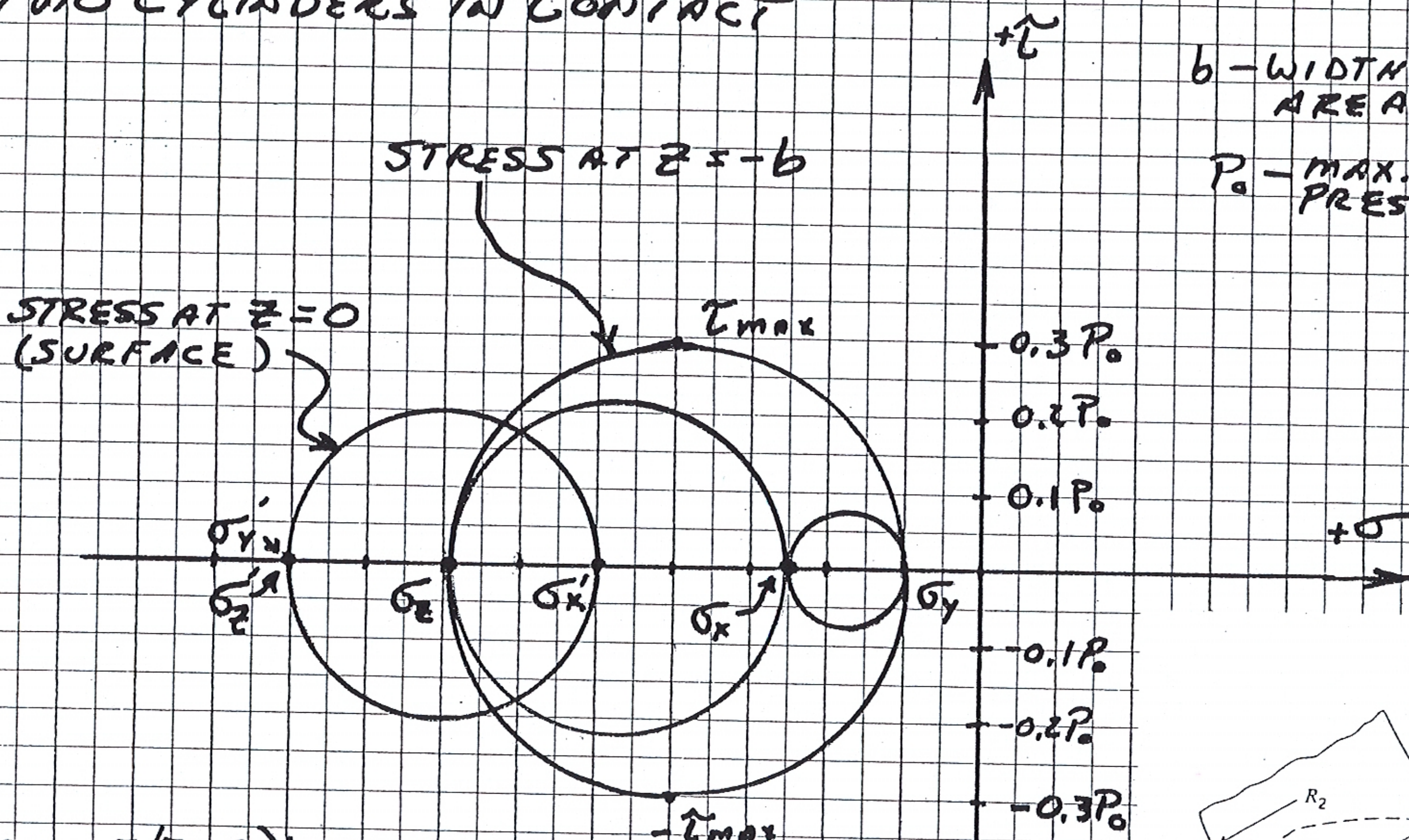
FIGURE 9.14
Contact pressure distribution.

MOHR'S CIRCLE

TWO CYLINDERS IN CONTACT

b – WIDTH OF CONTACT AREA A (RECTANGLE)

P_o – MAX. CONTACT PRESSURE



SURFACE (Z = 0):

$\sigma_z' = \sigma_y' = -P_o$

$\sigma_x' = -0.6\,P_o$

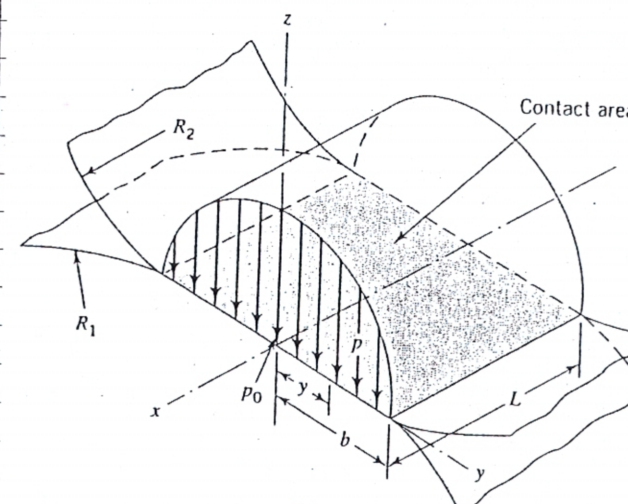
AT Z = –b:

$\sigma_z = -.7\,P_o$

$\sigma_x = -0.25\,P_o$

$\sigma_y = -0.1\,P_o$

$\tau_{max} = \pm 0.3\,P_o$



(b)

Two parallel cylinders

Bearing Failure Below Surface

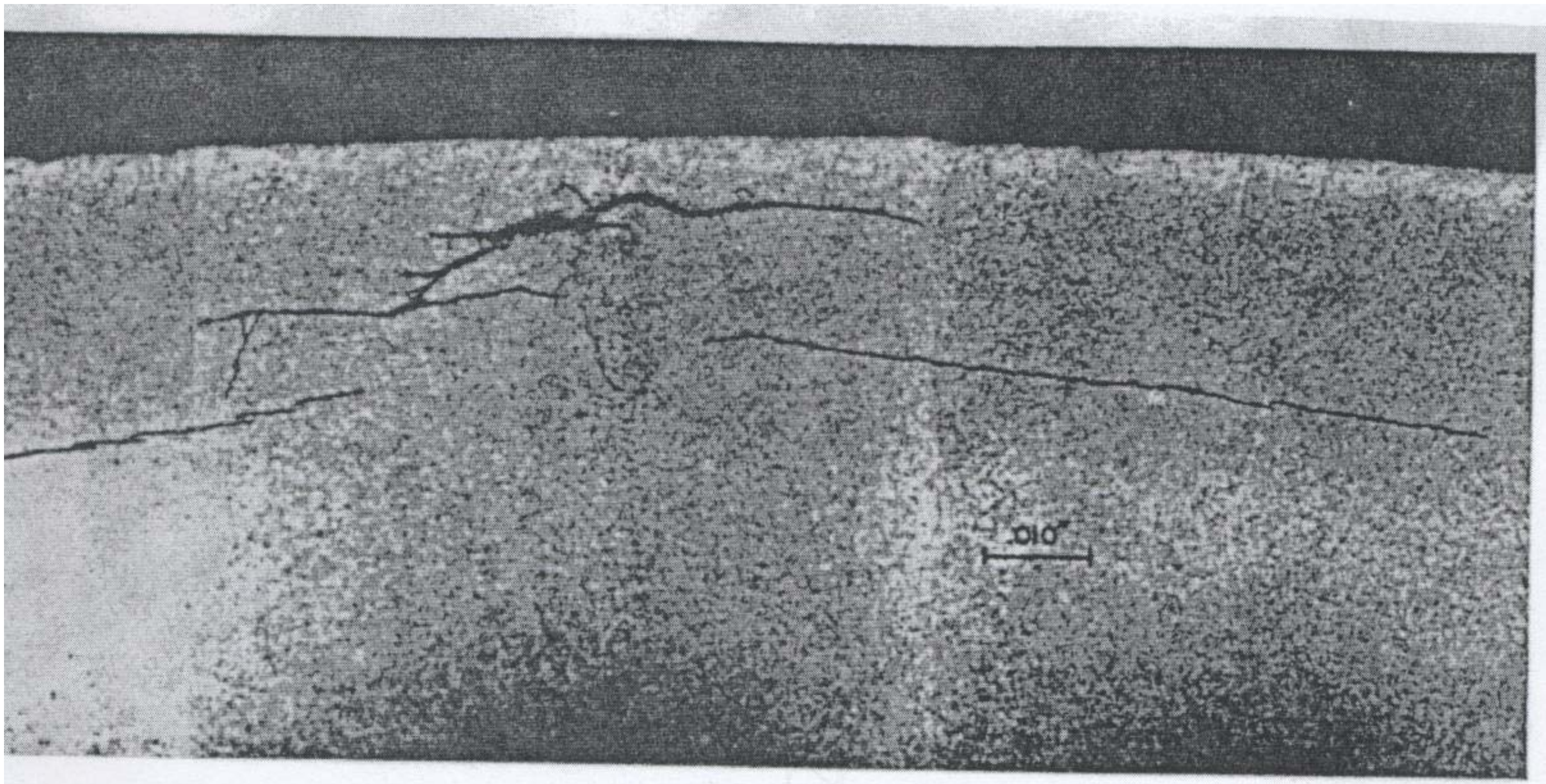


FIGURE 3

graph of a section perpendicular to a pin race showing a system of cracks. The bearing had run several hours with a mud containing sulfide; there were no cracks or pits on the race surface.

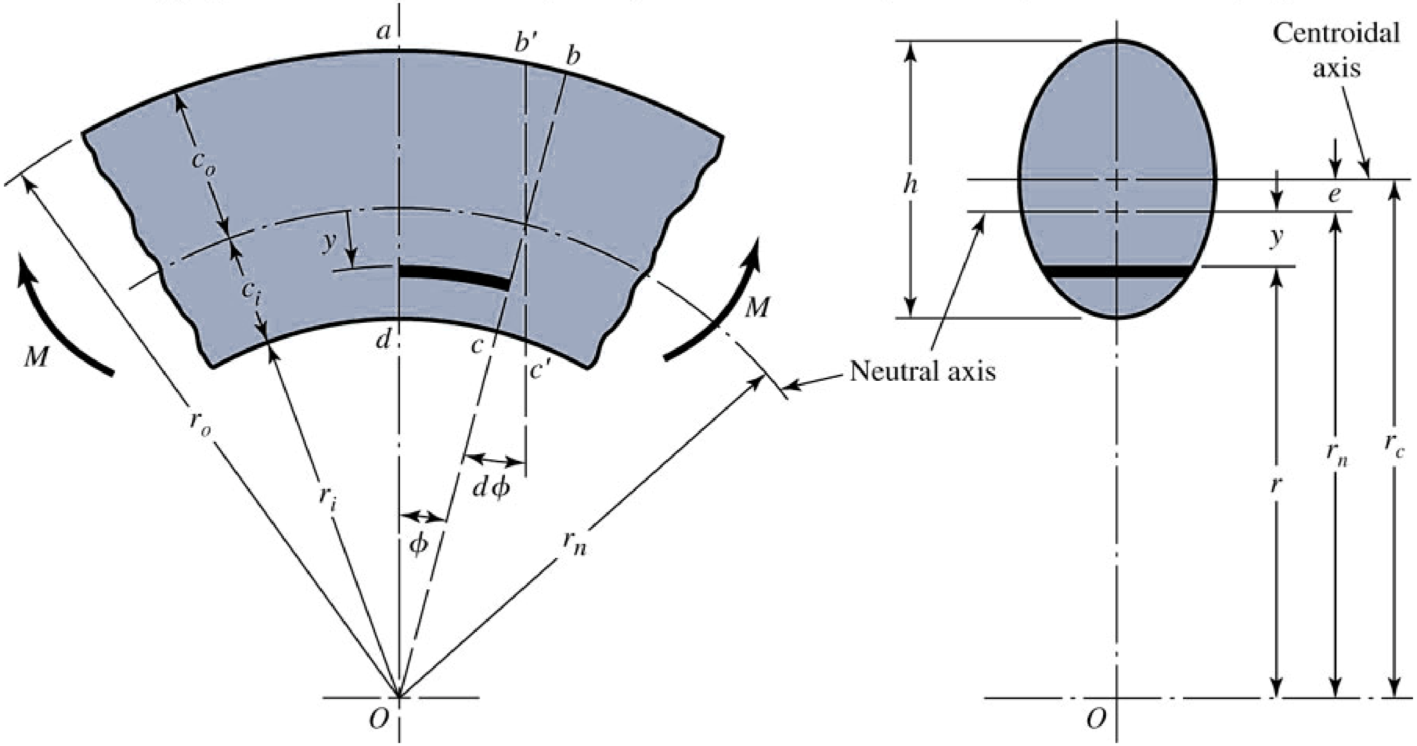
Contact stresses

- Most rolling members also tend to slide
  - Mating gear teeth
  - Cam and follower
  - Ball and roller bearings
- Resulting friction forces cause other stresses
  - Tangential normal and shear stresses
  - Superimposed on stresses caused by normal loading

Curved beams in bending

- Must use following assumptions
  - Cross section has axis of symmetry in a plane along the length of the beam
  - Plane cross sections remain plane after bending
  - Modulus of elasticity is same in tension and compression

Curved beams in bending, cont.



$$r_n = \frac{A}{\int \frac{dA}{r}}$$

$$\sigma = \frac{My}{Ae(r_n - y)}$$

$$\sigma_i = \frac{Mc_i}{Aer_i}$$

$$\sigma_o = -\frac{Mc_o}{Aer_o}$$